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ON THE HAUSDORFF MEASURE OF BROWNIAN PATHS IN THE PLANE

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1. *Introduction.* Ω will denote the space of all plane paths ω, so that ω is a short way of denoting the curve $z(t, \omega) = \{x(t, \omega), y(t, \omega)\}$ $(0 \leqslant t < +\infty)$. We assume that there is a probability measure μ defined on a Borel field $\mathscr{F}$ of (measurable) subsets of Ω, so that the system $(\Omega, \mathscr{F}, \mu)$ forms a mathematical model for Brownian paths in the plane. [For details of the definition of μ, see for example (9).] For

$$0 \leqslant a < b \leqslant +\infty, \quad \omega \in \Omega,$$

let $L(a, b; \omega)$ be the plane set of points $z(t, \omega)$ for $a \leqslant t \leqslant b$. Then with probability 1, $L(a, b; \omega)$ is a continuous curve in the plane. The object of the present note is to consider the measure of this point set $L(a, b; \omega)$.

The first step in this direction is due to Lévy (7) who proved that, with probability 1, the Lebesgue plane measure of $L(0, \infty; \omega)$ is zero. In (9), one of us considered the fractional dimension measure, that is the measure with respect to the function x^s $(0 < s \leqslant 2)$. It was proved that, again with probability 1, the measure with respect to each x^s $(0 < s < 2)$, is infinite: that is, the path has dimension 2 in the sense of Besicovitch. In (8), Lévy improved his zero Lebesgue measure result by proving that the measure of $L(0, 1; \omega)$ with respect to the function $x^2 \log\log 1/x$ is finite with probability 1. In fact Lévy proves this result for Brownian paths in n-dimensional Euclidean space $(n \geqslant 2)$, but states that he does not expect his result to be best possible for paths in the plane. He conjectured that in the plane case the measure of $L(0, 1; \omega)$ is finite with respect to the function $x^2 \log 1/x$.

$\phi(x)$ is called a *measure-function* if there is a $\delta > 0$ such that $\phi(x)$ is monotonic increasing and continuous for $0 < x < \delta$ and $\lim\limits_{x \to 0+} \phi(x) = 0$. For a set of points E in Euclidean space the Hausdorff measure of E with respect to $\phi(x)$, first defined in (4), is denoted by $\phi\text{-}m(E)$. Put

$$h_\alpha(x) = x^2(\log 1/x)^\alpha \quad (\alpha > 0).$$

Then $h_\alpha(x)$ is a measure function. In the present note we prove in § 3 that

$$h_1\text{-}m[L(0, 1; \omega)] < +\infty \tag{1}$$

with probability 1 (this establishes the conjecture of Lévy). The essential idea of our proof is to extend the results of Dvoretsky and Erdős (2) concerning random walks in the plane to the case of Brownian paths. Thus we consider how many squares of a covering mesh of squares side $1/n$ are entered by the path $L(0, 1; \omega)$.

To prove that $h_1(x)$ is the 'right' measure function for $L(0, 1; \omega)$ one should also show that

$$h_1\text{-}m[L(0, 1; \omega)] > 0 \tag{2}$$

with probability 1. Unfortunately, this turns out to be much more difficult to establish. We have a semi-heuristic proof of (2) which becomes impossibly complicated when one tries to make it rigorous. The essence of the difficulty is that one needs to show not only that *most* coverings of $L(0,1;\omega)$ are 'not too small' with respect to $h_1(x)$ but that it is impossible to find *any* covering which is arbitrarily small in the sense of h_1-measure.

It is worth mentioning at this stage that, if (2) could be established as well as (1), the law of zero or one would imply the existence of an absolute constant $\xi > 0$ such that, with probability 1, for all $0 < a < b < +\infty$

$$h_1\text{-}m[L(a,b;\omega)] = \xi(b-a). \tag{3}$$

Due to the connexion between Hausdorff measures and generalized capacities [see (5)], it is possible to obtain a lower bound for the measure of $L(a,b;\omega)$ by considering the capacity of $L(a,b;\omega)$ with respect to a suitable generalized capacity function. We push this method as far as possible and obtain, in § 4,

$$h_\alpha\text{-}m[L(0,1;\omega)] = \infty \quad \text{for all} \quad \alpha > 2, \tag{4}$$

with probability 1. This is clearly a long way from the desired result (2), but there are technical reasons why (4) is as much as one can hope to prove by using the method of capacities. These are discussed in (10).

2. *Preliminary results.* Suppose $\Lambda_n = \Lambda_n(0)$ is the lattice of points in the plane with coordinates $(q/n, r/n)$ where n is a positive integer and q, r are integers. If $z_0 = (x_0, y_0)$ is any point of the plane, $\Lambda_n(z_0)$ will denote the set of points of the form $(z - z_0)$ with $z \in \Lambda_n$. Then the set $\Lambda_n(z_0)$ defines a mesh of squares of side $1/n$ which have the points in $\Lambda_n(z_0)$ as their vertices: for convenience we suppose that each of these squares is closed on the left and open on the right. $S_0^{(n)}(z_0)$ will denote the square of this mesh which contains the origin O.

By the definition of $(\Omega, \mathscr{F}, \mu)$, the Brownian path starts from O with probability 1. Let

$$t_{r,n} = \frac{r}{n^2} \quad (n = 1, 2, \ldots; r = 0, 1, 2, \ldots), \tag{5}$$

$$Q_{r,n}(\omega) = z(t_{r,n}, \omega). \tag{6}$$

We start by looking at the path $L(0,1;\omega)$ only at the points $t_{r,n}$, and ask first how many squares of the mesh defined by $\Lambda_n(0)$ are required to cover all the points $Q_{r,n}(\omega)$ $(0 \leqslant r \leqslant n^2)$. In (2), Dvoretsky and Erdős obtained an asymptotic formula for the number of points entered in n steps by a plane random walk on a lattice. We now make some modifications in their method to fit the Brownian motion case.

Let $\gamma_{r,n}(z_0)$ denote the probability that none of the points $Q_{1,n}(\omega)$, $Q_{2,n}(\omega)$, $\ldots$, $Q_{r-1,n}(\omega)$ lie in the square $S_0^{(n)}(z_0)$—that is, the probability that the path $z(t,\omega)$ does not return to the 'origin square' at any of the points $t_{s,n}$ $(1 \leqslant s \leqslant r-1)$. Put

$$\gamma_{r,n} = n^2 \int_0^{1/n} \int_0^{1/n} \gamma_{r,n}(z_0)\, dx_0 dy_0. \tag{7}$$

If $z_0 \in S_0^{(n)}(0)$, we can think of $\gamma_{r,n}(z_0)$ as the probability that a Brownian path starting at z_0 will not be in the square $S_0^{(n)}(0)$ at any of the times $t_{s,n}$ $(1 \leqslant s \leqslant r-1)$. Thus, by

(7), $\gamma_{r,n}$ is the average value of $\gamma_{r,n}(z_0)$, assuming a uniform distribution for z_0. Now the measure μ in Ω is not changed if one simultaneously alters the time scale and the scale on the plane by multiplying t by a factor λ^2 and x, y each by a factor λ, when $\lambda > 0$. It follows that $\gamma_{r,n}$ is independent of n so we can write

$$\gamma_r = \gamma_{r,1} = \gamma_{r,2} = \dots = \gamma_{r,n}. \tag{8}$$

Our first object will be to obtain an asymptotic formula for γ_r.

It is clear that

$$1 = \gamma_0 \geqslant \gamma_1 \geqslant \dots \geqslant \gamma_n > 0. \tag{9}$$

Let the probability density function for the position of the point $Q_{r,n}(\omega)$ be $u_{r,n}(x,y)$: this is given by

$$u_{r,n}(x,y) = \frac{1}{2\pi t_{r,n}} \exp\left(\frac{-x^2-y^2}{2t_{r,n}}\right). \tag{10}$$

For any z_0, if $(x,y) \in S_0^{(n)}(z_0)$ we have $|x| \leqslant 1/n$, $|y| \leqslant 1/n$, and it follows from (5) and (10) that

$$\frac{n^2}{2\pi r} \geqslant u_{r,n}(x,y) \geqslant \frac{n^2}{2\pi r}\left(1-\frac{1}{r}\right), \tag{11}$$

for $r = 1, 2, \dots$. Thus for any z_0, if $w_{r,n}(z_0)$ denotes the probability that $Q_{r,n}(\omega)$ lies in $S_0^{(n)}(z_0)$, we have

$$\frac{1}{2\pi r} \geqslant w_{r,n}(z_0) \geqslant \frac{1}{2\pi r}\left(1-\frac{1}{r}\right). \tag{12}$$

Now we can classify the paths according to the last of the points $Q_{r,n}(\omega)$ $(r = 0, 1, \dots, k)$ which lie in the square $S_0^{(n)}(z_0)$. This gives

$$\gamma_{k,n}(z_0) + \sum_{r=1}^{k} \int_{S_0^{(n)}(z_0)} u_{r,n}(z)\, \gamma_{k-r,n}(z+z_0)\, dz = 1; \tag{13}$$

because if the path is at $z \in S_0^{(n)}(z_0)$ at time $t_{r,n}$, the probability that it will not return at any of the times $t_{s,n}$ $(r+1 \leqslant s \leqslant k)$, is the same as the probability that a path starting at O will not return to $S_0^{(n)}(z+z_0)$ at any of the times $t_{i,n}$ $(1 \leqslant i \leqslant k-r)$. Now $\gamma_{r,n}(z)$ is clearly periodic in x and y with period $1/n$ in x and $1/n$ in y. It follows from (7) and (8) that

$$\int_{S_0^{(n)}(z_0)} \gamma_{k-r,n}(z+z_0)\, dz = \frac{1}{n^2}\gamma_{k-r}. \tag{14}$$

Applying the lower bound in (11) gives

$$\gamma_{k,n}(z_0) + \frac{1}{2\pi} \sum_{r=1}^{k} \frac{1}{r}\left(1-\frac{1}{r}\right)\gamma_{k-r} \leqslant 1. \tag{15}$$

Averaging over z_0, and applying (9) leads to

$$\gamma_k\left\{\frac{1}{2\pi}\sum_{r=1}^{k}\frac{1}{r}\left(1-\frac{1}{r}\right)+1\right\} \leqslant 1,$$

or

$$\gamma_k \leqslant \frac{2\pi}{\log k} + O\left(\frac{1}{(\log k)^2}\right). \tag{16}$$

Using the upper bound of (11), we have for any z_0,

$$\gamma_{k,n}(z_0) + \sum_{r=1}^{k} \frac{1}{2\pi r}\gamma_{k-r,n} \geqslant 1; \tag{17}$$

so that, on averaging, if $1 \leqslant k_1 \leqslant k_2 \leqslant k$ we have, by (9),

$$\gamma_k + \gamma_{k-k_1} \sum_{r=1}^{k_1} \frac{1}{2\pi r} + \gamma_{k-k_2} \sum_{r=k_1+1}^{k_2} \frac{1}{2\pi r} + \sum_{r=k_2+1}^{k} \frac{1}{2\pi r} \geqslant 1.$$

With
$$k_1 = [\tfrac{1}{2}k], \quad k_2 = \left[k - \frac{k}{\log k}\right],$$

this gives
$$\gamma_k \geqslant \frac{2\pi}{\log k} + O\left(\frac{1}{(\log k)^2}\right),$$

which with (16) shows that
$$\gamma_{k_1} = \frac{2\pi}{\log k} + O\left(\frac{1}{(\log k)^2}\right). \tag{18}$$

If z_0, z_0' are any two points, subtracting (17) with z_0 replaced by z_0' from (15) and simplifying yields
$$\gamma_{k,n}(z_0) - \gamma_{k,n}(z_0') \leqslant \frac{1}{\log k} \sum_{r=1}^{\infty} \frac{1}{r^2}(1+o(1))$$
$$< \frac{\pi}{\log k}(1+o(1)).$$

Since γ_k is the average value of $\gamma_{k,n}(z_0)$ for all z_0, it follows from (18) that, for sufficiently large k, and any z_0,
$$\tfrac{1}{2}\gamma_k \leqslant \gamma_{k,n}(z_0) \leqslant 2\gamma_k. \tag{19}$$

Now consider the fixed lattice $\Lambda_n(0)$. Each point of the plane is in precisely one of the half-open squares of side $1/n$ defined by $\Lambda_n(0)$. The density distribution (10) therefore defines a density distribution relative to the mesh. That is, if we are only interested in the position of (x, y) relative to the square in which it lies, we obtain a density distribution $v_{r,n}(x, y)$ at $t = t_{r,n}$ given by

$$v_{r,n}(x, y) = \frac{1}{2\pi t_{r,n}} \sum_{s=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} \exp\left(\frac{-(x+s/n)^2 - (y+q/n)^2}{2t_{r,n}}\right)$$

for $0 \leqslant x < 1/n$, $0 \leqslant y < 1/n$ with $v_{r,n}(x, y) = 0$ for other points (x, y). A simple computation shows that $v_{r,n}(x, y)$ defines a distribution which, for large integers r, becomes very close to uniform in the square $S_0^{(n)}(0)$. We only require that

$$v_{r,n}(x, y) = n^2 \phi_r(x, y), \tag{20}$$

where
$$\phi_r(x, y) = 1 + o(1/r) \quad \text{as} \quad r \to \infty;$$

for (x, y) in $S_0^{(n)}(0)$.

Still considering the fixed lattice $\Lambda_n(0)$, let $\tau_{k,n}$ be the probability that $Q_{k,n}(\omega)$ lies in a new square of side $1/n$: that is, that $Q_{k,n}(\omega)$ lies in a square which contains none of the points $Q_{0,n}(\omega)$, $Q_{1,n}(\omega)$, ..., $Q_{k-1,n}(\omega)$. Now a change of scale again shows that $\tau_{k,n}$ is independent of n, so we will denote it merely by τ_k. We can think of $z(t_{r,n}, \omega)$ as the result of adding together r independent random vectors each distributed like $z(1/n^2, \omega)$. Hence we can apply the argument on p. 354 of (2) to deduce that, if the distribution of $Q_{k,n}(\omega)$ relative to the square in which it lies were uniform, then we would have $\tau_k = \gamma_k$. However, the distribution is almost uniform so that, by (20),

$$\tau_k = \gamma_k\{1+O(1/k)\};$$

or

$$\tau_k = \frac{2\pi}{\log k} + O\left(\frac{1}{(\log k)^2}\right) \tag{21}$$

on substituting (18).

Now let $N_n(\omega)$ be the number of squares of the lattice $\Lambda_n(0)$ which are entered by at least one of the points $Q_{k,n}(\omega)$ $(0 \leqslant k \leqslant n^2)$. It is clear that

$$\mathscr{E}\{N_n(\omega)\} = \sum_{k=0}^{n^2} \tau_k.$$

Substituting the estimate (21) gives

$$\mathscr{E}\{N_n(\omega)\} = \frac{\pi n^2}{\log n} + O\left(\frac{n^2 \log\log n}{(\log n)^2}\right). \tag{22}$$

By using the relation (19), the methods of (2) can be modified without difficulty to estimate the variance of $N_n(\omega)$. This gives

$$\sigma^2\{N_n(\omega)\} = O\left(\frac{n^2 \log\log n}{(\log n)^3}\right). \tag{23}$$

The strong law for $N_n(\omega)$ does not follow immediately by the methods of (2), though it can be proved by a more detailed investigation. We do not do this in the present note as it is not required—the strong law for a subsequence is sufficient.

LEMMA 1. *If* $n_s = 2^{2^s}$ $(s = 1, 2, \ldots)$, *then*

$$\mu\left\{\omega\colon \lim_{s\to\infty}\left[N_{n_s}(\omega)\frac{\log n_s}{\pi n_s^2}\right] = 1\right\} = 1.$$

Proof. Chebyshev's inequality applied to (22) and (23) gives

$$\mu\left\{\omega\colon \left|N_n(\omega) - \frac{\pi n^2}{\log n}\right| > \epsilon\frac{n^2}{\log n}\right\} = O\left(\frac{\log\log n}{\log n}\right).$$

It follows that, for any $\epsilon > 0$,

$$\Sigma\mu\left\{\omega\colon \left|N_{n_s}(\omega) - \frac{\pi n_s^2}{\log n_s}\right| > \epsilon\frac{n_s^2}{\log n_s}\right\}$$

converges. An application of the Borel–Cantelli lemma is now sufficient to give the required result.

We now need a lemma which shows that any square of the mesh defined by $\Lambda_n(z_0)$ which contains at least one point $Q_{r,n}$ $(0 \leqslant r \leqslant n^2)$, is likely to contain very many of them.

LEMMA 2. *Suppose* $0 < \delta \leqslant \frac{1}{2}$; *for a given path* ω, *let the number of the points* $z(t_{r,n}, \omega)$ *with* $1 \leqslant r \leqslant n^2$ *which lie in the square* $S_0^{(n)}(z_0)$ *be* $Y_n(z_0; \omega)$. *Then, for any* z_0,

$$\mu\{\omega\colon Y_n(z_0; \omega) < (\log n)^{\delta}\} = \frac{\pi}{(\log n)^{1-\delta}} + O\left(\frac{\log\log n}{\log n}\right).$$

Proof. The point $z(t_{0,n}, \omega) = 0$ lies in $S_0^{(n)}(z_0)$. Let $0 = r_0 < r_1 < r_2 < \ldots$ be the sequence of integers, defined by the path ω with probability 1, such that $z(t_{r,n}, \omega)$ lies in $S_0^{(n)}(z_0)$ for $r = r_1, r_2, \ldots$; but not for $r_{i-1} < r < r_i$ $(i = 1, 2, \ldots)$. Now it follows from (10) that the distribution of the point $z(t_{r,n}, \omega)$ in $S_0^{(n)}(z_0)$, given that it lies in this square, is very nearly uniform for large r; in fact, it differs from the uniform distribution by a factor ϕ where

$$|\phi - 1| < \frac{1}{r}.$$

Hence, since $r_i \geqslant i$, for each positive integer k

$$\{1-(1/i)\}\gamma_k < \mu\{\omega\colon r_i - r_{i-1} > k\} < \{1+(1/i)\}\gamma_k. \tag{24}$$

Put $s = [(\log n)^\delta]$. Then $Y_n(z_0; \omega) < (\log n)^\delta$ if, and only if, $r_s > n^2$. Now if $Y_n(z_0; \omega) \leqslant s$, then $r_i - r_{i-1} > \{n^2/(\log n)^\delta\}$ for at least one i $(1 \leqslant i \leqslant s)$; so that

$$\mu\{\omega\colon Y_n(z_0; \omega) \leqslant s\} \leqslant \sum_{i=1}^{s} \mu\left\{\omega\colon r_i - r_{i-1} > \frac{n^2}{(\log n)^\delta}\right\}.$$

Using (24) and (18), this yields

$$\mu\{\omega\colon Y_n(z_0; \omega) \leqslant s\} \leqslant \frac{\pi}{(\log n)^{1-\delta}} + O\left(\frac{\log\log n}{\log n}\right). \tag{25}$$

In the other direction, notice that if there is one integer i_0 $(1 \leqslant i_0 \leqslant s)$ such that $r_{i_0} - r_{i_0-1} > n^2$ while $r_i - r_{i-1} \leqslant n^2$ for $i \neq i_0$, $1 \leqslant i \leqslant s$, then $Y_n(z_0; \omega) < s$. For different integers i_0 these events are mutually exclusive. Hence

$$\mu\{\omega\colon Y_n(z_0; \omega) < s\} \geqslant \sum_{i_0=1}^{s} \mu\{\omega\colon r_{i_0} - r_{i_0-1} > n^2 \text{ but } r_i - r_{i-1} \leqslant n^2 \text{ for } i \neq i_0, 1 \leqslant i \leqslant s\}.$$

Since the inequality (24) was true for any z_0, we have

$$\begin{aligned}\mu\{\omega\colon Y_n(z_0; \omega) < s\} &\geqslant \sum_{i_0=1}^{s} \left(1 - \frac{1}{i_0}\right)\gamma_{n^2} \prod_{i=1}^{s}\left\{1 - \left(1 + \frac{1}{i}\right)\gamma_{n^2}\right\} \\ &\geqslant \frac{\pi}{(\log n)^{1-\delta}} + O\left(\frac{\log\log n}{\log n}\right)\end{aligned}$$

on using (18). This, together with (25) clearly establishes the lemma.

LEMMA 3. *Let $T_\delta(n; \omega)$ $0 < \delta \leqslant \frac{1}{2}$, be the number of squares of the mesh defined by $\Lambda_n(0)$ which contains at least one $z(t_{r,n}, \omega)$, but less than $(\log n)^\delta$ of the points $z(t_{r,n}, \omega)$ for $0 \leqslant r \leqslant n^2$. Then*

$$\mathscr{E}\{T_\delta(n; \omega)\} = \frac{\pi^2 n^2}{(\log n)^{2-\delta}} + O\left(\frac{n^2 \log\log n}{(\log n)^2}\right).$$

Proof. Suppose $\lambda(r)$, $0 \leqslant r \leqslant n^2$, is the probability that (i) $z(t_{r,n}, \omega)$ lies in a square of the mesh not containing any of the points $z(t_{i,n}, \omega)$ for $0 \leqslant i < r$, and in addition (ii) fewer than $(\log n)^\delta$ of the points $z(t_{i,n}, \omega)$, $r \leqslant i \leqslant n^2$ lie in the same square as $z(t_{r,n}, \omega)$. Clearly

$$\mathscr{E}\{T_\delta(n; \omega)\} = \sum_{r=0}^{n^2} \lambda(r). \tag{26}$$

Now $$\lambda(r) \geqslant \tau_r \inf_{z_0 \in S_0^{(n)}(0)} \mu\{\omega\colon Y_n(z_0; \omega) < (\log n)^\delta\},$$

since $Y_n(z_0;\omega) < (\log n)^\delta$ has the same probability as the event that in the n^2 steps $z(t_{i,n},\omega)$ $(r \leqslant i \leqslant r+n^2)$, starting from $z_0 = z(t_{r,n},\omega) \in S_0^{(n)}(0)$ there are less than $(\log n)^\delta$ returns to $S_0^{(n)}(0)$. From Lemma 2 and (21) it follows that

$$\mathscr{E}\{T_\delta(n;\omega)\} \geqslant \left\{\frac{\pi}{(\log n)^{1-\delta}} + O\left(\frac{\log\log n}{\log n}\right)\right\}\frac{\pi n^2}{\log n}\left(1+O\frac{1}{(\log n)}\right),$$

so that
$$\mathscr{E}\{T_\delta(n;\omega)\} \geqslant \frac{\pi n^2}{(\log n)^{2-\delta}} + O\left(\frac{n^2\log\log n}{(\log n)^2}\right). \tag{27}$$

Let $p(z_0)$ be the probability that in $[n^2/(\log n)^2]$ steps there are fewer than $(\log n)^\delta$ returns to $S_0^{(n)}(z_0)$. The argument of Lemma 2 shows that

$$p(z_0) = \frac{\pi}{(\log n)^{1-\delta}} + O\left(\frac{\log\log n}{\log n}\right). \tag{28}$$

If
$$0 \leqslant r \leqslant n^2 - \left[\frac{n^2}{(\log n)^2}\right] \quad (= N, \text{ say})$$

it is clear that
$$\lambda(r) \leqslant \tau_r \sup_{z_0 \in S_0^{(n)}(0)} p(z_0),$$

while for $N < r \leqslant n^2$,
$$\lambda(r) \leqslant \tau_r.$$
Hence, by (28) and (26), we have

$$\begin{aligned}\mathscr{E}\{T_\delta(n;\omega)\} &\leqslant \sum_{r=0}^{N}\lambda(r) + \sum_{r=N}^{n^2}\tau_r \\ &\leqslant \left[\frac{\pi}{(\log n)^{1-\delta}} + O\left(\frac{\log\log n}{\log n}\right)\right]\sum_{r=0}^{n^2}\tau_r + O\left(\frac{n^2}{(\log n)^3}\right) \\ &= \frac{\pi^2 n^2}{(\log n)^{2-\delta}} + O\left(\frac{n^2\log\log n}{(\log n)^2}\right).\end{aligned}$$

This, together with (27), establishes the lemma.

LEMMA 4. *Let S' be a fixed square of side $1/n$ whose distance from the origin is less than ρ/n where $\rho < k^{\frac{1}{2}}$, k a positive integer; and suppose $F_k(S')$ is the event that none of the points $Q_{r,n}(\omega)$, $0 \leqslant r \leqslant k$ lies in the square S'. Then there is a real constant c_1 such that*

$$\mu\{F_k(S')\} < \frac{c_1(\log\rho+1)}{\log k}.$$

Proof. The result corresponding to this lemma was obtained for plane random walks in (3). Modifications to the methods of (3), similar to those used in the present paper in obtaining the estimate (18) for γ_k, are sufficient to prove the lemma, so we will omit the details.

We need to consider not only the points $Q_{r,n}(\omega)$ of the path $L(0,1;\omega)$ but also the points $z(t,\omega)$ for $t_{r-1,n} \leqslant t \leqslant t_{r,n}$ $(r = 1, 2, \ldots)$. For this purpose we shall need estimates for the largest variation $|z(t,\omega) - z(t_{r-1,n},\omega)|$ for $t_{r-1,n} \leqslant t \leqslant t_{r,n}$. The next two lemmas give the results we need.

LEMMA 5. *If $\rho > 1$, then there is a finite constant c_3 such that*

$$\mu\{\omega\colon \sup_{t_{r,n}\leqslant\tau\leqslant t_{r+1,n}} |z(\tau,\omega)-z(t_r,\omega)| > (\rho/n)\} < \frac{c_3}{\rho}\exp\left(-\frac{\rho^2}{2}\right).$$

This result is well known; for example, it is an immediate deduction from Lemma 4 of (9).

LEMMA 6. *For a fixed path ω in Ω, there is probability 1 that*

$$\limsup_{n\to\infty}\left\{\sup_{0\leqslant r\leqslant n^2}\left(\frac{n}{\sqrt{(2\log n)}}\sup_{t_{r,n}\leqslant\tau\leqslant t_{r+1,n}}|z(\tau,\omega)-z(t_{r,n},\omega)|\right)\right\} = 1.$$

This is the two-dimensional form of a result which is well known in one dimension: a proof of the one-dimensional case may be found in (6), p. 152.

3. *Covering the whole path $L(0,1;\omega)$.* The result of Lemma 1 shows how many squares of side $1/n_s$ of the mesh defined by Λ_{n_s} are needed to cover the n_s^2 discrete points $Q_{r,n_s}(\omega)$ $(0\leqslant r\leqslant n_s^2)$. Our object now will be to show that the whole path $L(0,1;\omega)$ can be covered without increasing the number of squares needed by a factor larger than one. We have seen already in Lemma 3 that most of the squares of side $1/n$ which contain one of the points $Q_{r,n}(\omega)$ will contain a very large number. From this we will deduce that the path is highly concentrated on the little squares entered; that is, that most of the squares which contain one of the points $Q_{r,n}(\omega)$ are such that all the squares of the mesh nearby also contain at least one of the points $Q_{r,n}(\omega)$. By this means we can show that it is possible to expand each of the little squares containing a point $Q_{r,n}(\omega)$ by a suitable large factor, without increasing significantly the total number of little squares of side $1/n$ which are included in the covering. The expansion factor will be chosen large enough to ensure that all of $L(t_{r,n},t_{r+1,n};\omega)$ will be included in the larger region about the square which contains $Q_{r,n}(\omega)$.

We need to consider three types of 'bad' points, i.e. points which would spoil our covering estimate.

(i) Those integers r for which there is a very large displacement between $t_{r,n}$ and $t_{r+1,n}$. When n is large, Lemma 6 gives an adequate upper bound to the size of the 'large' displacements. It is convenient to call the point $Q_{r,n}(\omega)$ bad in this sense if

$$\sup_{t_{r,n}\leqslant\tau\leqslant t_{r+1,n}}|z(\tau,\omega)-z(t_r,\omega)| > \frac{\rho_n}{n}, \tag{29}$$

where $\rho_n = \log\log n$. The number of integers r $(0\leqslant r\leqslant n^2)$, for which (29) is satisfied will be denoted by $B_1(n,\omega)$.

(ii) Those integers r such that $Q_{r,n}(\omega)$ lies in a square S of the mesh $\Lambda_n(0)$ such that S contains fewer than $(\log n)^{\frac{1}{2}}$ of the points $Q_{r,n}(\omega)$. The number of integers r which are bad in this sense will be denoted by $B_2(n,\omega)$. In fact $B_2(n,\omega) = T_{\frac{1}{2}}(n,\omega)$ as defined in Lemma 3.

(iii) Those points which are not bad in sense (ii) but, nevertheless, are such that there is at least one square of the mesh whose distance from S is less than $2\log\log n/n$, but which contains none of the points $Q_{r,n}(\omega)$. The number of integers r which are bad in this sense will be denoted by $B_3(n,\omega)$.

For any positive integer n, let $L_n(\omega)$ denote the number of squares of the mesh $\Lambda_n(0)$ which contain at least one point of $L(0,1;\omega)$. $N_n(\omega)$ was the number of squares which contain at least one of the points $Q_{r,n}(\omega)$ $(0 \leqslant r \leqslant n^2)$. Let us now obtain $R_n(\omega)$ in the following way. If a little square S of side $1/n$ contains at least one of the points $Q_{r,n}(\omega)$ $(0 \leqslant r \leqslant n^2)$, then take all the squares of the mesh which are within $\log\log n/n$ of S. In addition, if S contains a point $Q_{r,n}(\omega)$ which is bad in sense (i), take the squares of the mesh which are within $(2/n)(\log n)^{\frac{1}{2}}$ of S. $R_n(\omega)$ is the total number of squares obtained in this way. It follows from Lemma 6, that for sufficiently large n, the set of squares obtained must cover all of $L(0,1;\omega)$. Hence for large enough n

$$N_n(\omega) \leqslant L_n(\omega) \leqslant R_n(\omega). \tag{30}$$

Thus to obtain an estimate for $L_n(\omega)$ it is sufficient to show that

$$\{R_n(\omega)\}/\{N_n(\omega)\} \to 1$$

and use Lemma 1. In obtaining an estimate for the measure of $L(0,1;\omega)$ it is sufficient to obtain arbitrarily fine coverings of bounded 'extent'. Hence it will be sufficient to show that, with probability 1, if $n_s = 2^{2^s}$

$$\lim_{s\to\infty} \frac{R_{n_s}(\omega)}{N_{n_s}(\omega)} = 1. \tag{31}$$

We need estimates of the numbers $B_i(n;\omega)$ $(i = 1,2,3)$ of bad points. It follows immediately from Lemma 5 that,† for large n,

$$\mathscr{E}\{B_1(n,\omega)\} < \frac{n^2}{(\log n)^{10}}, \tag{32}$$

Hence
$$\mu\left\{\omega\colon B_1(n,\omega) > \frac{n^2}{(\log n)^5}\right\} < \frac{1}{(\log n)^5}.$$

An application of the Borel–Cantelli lemma shows that there exists, with probability 1, an integer s_1 such that

$$B_1(n_s,\omega) \leqslant \frac{n_s^2}{(\log n_s)^5} \quad \text{for} \quad s \geqslant s_1. \tag{33}$$

Lemma 3 already gives a good estimate for $\mathscr{E}\{B_2(n,\omega)\}$. This leads to

$$\mu\left\{\omega\colon B_2(n,\omega) > \frac{n^2}{(\log n)^{\frac{5}{4}}}\right\} < \frac{c_4}{(\log n)^{\frac{1}{4}}}$$

for a suitable positive constant c_4. Again the Borel–Cantelli lemma shows that there exists, with probability 1 an integer s_2 such that

$$B_2(n_s,\omega) \leqslant \frac{n_s^2}{(\log n_s)^{\frac{5}{4}}} \quad \text{for} \quad s \geqslant s_2. \tag{34}$$

Suppose now S is a square of the mesh which contains at least $(\log n)^{\frac{1}{2}}$ of the points $Q_{r,n}(\omega)$ $(0 \leqslant r \leqslant n)$. Let $q_n(S)$ be the conditional probability that there exists another square of the mesh within $2\log\log n/n$ of S with none of the points $Q_{r,n}(\omega)$ in it. We need first to estimate $q_n(S)$. Suppose i_0 is the first integer r for which $Q_{r,n}(\omega)$ lies in S.

† Much more than this is true, but (32) is sufficient for our present purposes.

Consider a fixed square S' within $2\log\log n/n$ of S. By Lemma 4, the probability p_1 that none of the points $Q_{r,n}(\omega)$ for

$$i_0 \leqslant r \leqslant i_0 + (\log n)^{\frac{1}{4}}$$

lie in S' satisfies

$$p_1 < c_5 \frac{\log\log\log n}{\log\log n}.$$

Let i_1 be the first integer $r > i_0 + (\log n)^{\frac{1}{4}}$ such that $Q_{r,n}(\omega)$ is in S again. Then the probability of not entering S' for $i_1 \leqslant r \leqslant i_1 + (\log n)^{\frac{1}{4}}$ is again

$$p_2 < c_5 \frac{\log\log\log n}{\log\log n}.$$

Repeat the process, obtaining

$$i_1 < i_2 < \dots < i_t < n^2.$$

Since S contains at least $(\log n)^{\frac{1}{2}}$ points $Q_{r,n}(\omega)$ and there cannot be more than $(\log n)^{\frac{1}{4}}$ in any of the ranges (i_k, i_{k+1}) it follows that $t \geqslant (\log n)^{\frac{1}{4}}$. Hence the probability that the square S' contains no point $Q_{r,n}(\omega)$ satisfies

$$p < \left(c_5 \frac{\log\log\log n}{\log\log n}\right)^{(\log n)^{\frac{1}{4}}}.$$

But there are not more than $15(\log\log n)^2$ squares of side $1/n$ which are within $2\log\log n/n$ of S. Hence the required probability q_n satisfies

$$q_n(S) < 15(\log\log n)^2 \left[c_5 \frac{\log\log\log n}{\log\log n}\right]^{(\log n)^{\frac{1}{4}}}.$$

This implies, *a fortiori* for large n,

$$q_n(S) < \exp\left(-(\log n)^{\frac{1}{4}}\right).$$

Since there are not more than n^2 squares, it follows immediately that

$$\mathscr{E}\{B_3(n,\omega)\} < n^2 \exp\left(-(\log n)^{\frac{1}{4}}\right). \tag{35}$$

A further application of the Borel–Cantelli lemma shows that there exists with probability 1 an integer s_3 such that

$$B_3(n_s, \omega) \leqslant \frac{n_s^2}{(\log n_s)^2} \quad \text{for} \quad s \geqslant s_3. \tag{36}$$

Finally, by Lemma 6, there exists with probability 1 an integer s_4 such that

$$\sup_{0 \leqslant r \leqslant n^2} \Big(\sup_{t_{r,n} \leqslant \tau \leqslant t_{r+1,n}} |z(\tau,\omega) - z(t_{r,n},\omega)|\Big) \leqslant \frac{2\sqrt{\log n}}{n} \tag{37}$$

for all $n \geqslant n_{s_4}$.

Now, if $K = \max(s_1, s_2, s_3, s_4)$, it is easy to deduce from (33), (34) and (36) that the number of extra little squares we have included by our procedure satisfies

$$R_{n_s}(\omega) - N_{n_s}(\omega) < \frac{n_s^2}{(\log n_s)^{\frac{9}{8}}} \tag{38}$$

for all $s \geqslant K$. Further (37) ensures that the procedure includes sufficient little squares to cover the whole path. Lemma 1 together with (38) establishes (31) which in turn implies

LEMMA 7. *Let $L_n(\omega)$ denote the number of squares of side $1/n$ of the mesh $\Lambda_n(0)$ which are entered by at least one point of $L(0, 1; \omega)$. Then, if $n_s = 2^{2^s}$ $(s = 1, 2, \ldots)$,*

$$\mu\left\{\omega: \lim_{s\to\infty}\left(L_{n_s}(\omega)\frac{\log n_s}{\pi n_s^2}\right) = 1\right\} = 1.$$

We can immediately deduce

THEOREM 1. *If $h_1(x) = x^2 \log 1/x$, then there is probability 1 that*

$$h_1\text{-}m\{L(0, 1; \omega)\} \leqslant 2\pi.$$

Proof. The diameter of a square of side $1/n_s$ is $\sqrt{2}/n_s$. For any $\epsilon > 0$, we have just shown that for large s it is possible to cover $L(0, 1; \omega)$ with less than

$$\frac{(\pi+\epsilon)\, n_s^2}{\log n_s} \text{ squares of diameter } \frac{\sqrt{2}}{n_s}.$$

This gives an h_1-measure covering by arbitrarily small squares of extent less than $2(\pi+\epsilon)$. The theorem follows immediately since ϵ is arbitrary.

Remark. It is clear that the constant in Theorem 1 is not best possible. If one covers the path with a mesh of hexagons instead of the mesh of squares we have used it can be proved that

$$h_1\text{-}m\{L(0, 1; \omega)\} \leqslant \frac{8\pi}{3\sqrt{3}};$$

however, even this result may not be best possible.

4. *A lower bound for the measure*. In proving that the h_1-measure of $L(0, 1; \omega)$ is finite we only had to show that it is possible to give a covering of the path of bounded extent using figures of arbitrarily small diameter. However, if we are to succeed in showing that the h_1-measure is positive then we must show that *no* covering is possible by a collection of convex sets of arbitrarily small total extent. This is much more difficult. Besicovitch (1) showed that in order to prove positive measure it is sufficient only to consider coverings by squares belonging to one of the meshes $\Lambda_n(0)$, $n = 2^k$ $(k = 1, 2, \ldots)$. In fact for the particular measure function $h_1(x)$ it can be shown that it is sufficient to consider coverings by squares which belong to one of the meshes Λ_{n_s} $(s = 1, 2, \ldots)$. [This follows from the fact that, in a certain sense, h_1-measure differs only slightly from Lebesgue measure.] However, one still must allow for the possibility that the covering set may consist of squares of widely different sizes: it is this which causes the difficulty. In § 2 we saw that the path was fairly thickly concentrated in the regions which it enters. We would now need to prove that this concentration is not 'too thick', i.e. that not too large a proportion of the squares of the mesh side $1/n_k$ contained in a single square of side $1/n_t$ are entered when k is much greater than t. We have not succeeded in completing this proof rigorously.

For the remainder of this section let us push the technique of using generalized capacity as far as possible and see what can be deduced. Suppose $\Phi(r)$ is any continuous function defined for $r > 0$ such that $\lim_{x\to 0+} \Phi(x) = +\infty$. For a closed set E in the plane, consider the class of all measures ν defined on E such that $\nu(E) = 1$. We say that the Φ-capacity of E denoted by $C^{(\Phi)}(E)$ is zero if

$$\int_E \int_E \Phi(|x-y|)\, d\nu(x)\, d\nu(y)$$

fails to converge for all measures ν; on the other hand, if we can find a measure ν such that

$$\int_E \times \int_E \Phi(|x-y|)\, d\nu(x)\, d\nu(y) < +\infty \tag{39}$$

we say that $C^{(\Phi)}(E) > 0$.†

h-measure and Φ-capacity are connected by the following result due to Kematani (5).

LEMMA 8. *If $h(x)$ and $\Phi(x)$ are respectively a measure function and capacity function such that $\Phi(x) = 1/h(x)$ and E is any closed set with h-$m(E)$ finite, then $C^{(\Phi)}(E) = 0$.*

LEMMA 9. *For any $k > 0$, $t > 0$,*

$$\mu\{\omega\colon |z(t,\omega)| < k\} = \frac{1}{t}\int_0^k r \exp\left(-\frac{r^2}{2t}\right) dr.$$

This is proved in (9).

THEOREM 2. *If $h_\alpha(x) = x^2(\log 1/x)^\alpha$, then there is probability 1 that*

$$h_\alpha\text{-}m\{L(0,1;\omega)\} = +\infty$$

for every $\alpha > 2$.

Proof. As far as Hausdorff measure is concerned only covering sets of small diameter are relevant. Hence if

$$\phi_\alpha(x) = \begin{cases} h_\alpha(x) & (0 < x \leqslant \frac{1}{2}) \\ \frac{1}{4}(\log 2)^\alpha = \lambda_\alpha & (x \geqslant \frac{1}{2}) \end{cases} \quad (\alpha > 0), \tag{40}$$

then h_α-$m(E) = \phi_\alpha$-$m(E)$ for all E. Consider

$$I(t) = \int_\Omega \frac{d\omega}{\phi_\alpha(|z(t,\omega)|)} = \frac{1}{t}\int_0^{\frac{1}{2}} \frac{\exp\{-(r^2/2t)\}}{r\{\log(1/t)\}^\alpha}\, dr + \frac{1}{\lambda_\alpha t}\int_{\frac{1}{2}}^\infty r\exp\{-(r^2/2t)\}\, dr,$$

using (20) and Lemma 9. By making change of variable $r = t^{\frac{1}{2}}x$ we obtain, provided $0 < t < \frac{1}{16}$,

$$\begin{aligned} I(t) &= \frac{1}{t}\int_0^{\frac{1}{2}t^{-\frac{1}{2}}} \frac{\exp(-\frac{1}{2}x^2)}{x\{\log(1/xt^{\frac{1}{2}})\}^\alpha}\, dx + \lambda_\alpha^{-1}\int_{\frac{1}{2}t^{-\frac{1}{2}}}^\infty x\exp(-\tfrac{1}{2}x^2)\, dx \\ &= \frac{1}{t}\left\{\int_0^1 + \int_1^{t^{-\frac{1}{4}}} + \int_{t^{-\frac{1}{4}}}^{\frac{1}{2}t^{-\frac{1}{2}}}\right\} \frac{\exp(-\frac{1}{2}x^2)}{x(\log 1/xt^{\frac{1}{2}})^\alpha}\, dx + \lambda_\alpha^{-1}\exp(-\tfrac{1}{4}t) \\ &= I_1 + I_2 + I_3 + \lambda_\alpha^{-1}\exp(-\tfrac{1}{4}t), \quad \text{say.} \end{aligned}$$

† An actual numerical value for $C^{(\Phi)}(E)$ can be defined in terms of the infimum of the set of values of the integral occurring in (39). However, this is unimportant.

Now

$$I_1(t) = \frac{1}{t}\int_0^1 \frac{\exp(-\frac{1}{2}x^2)}{x\{\log(1/x)+\frac{1}{2}\log 1/t\}^\alpha}\,dx$$

$$< \frac{c_6}{t(\log 1/t)^{\frac{1}{2}\alpha}},$$

for suitable $c_6 > 0$.

$$I_2(t) = \frac{1}{t}\int_1^{t^{-\frac{1}{4}}} \frac{\exp(-\frac{1}{2}x^2)}{x\{\log(1/x)+\frac{1}{2}\log 1/t\}^\alpha}\,dx$$

$$< \frac{1}{t(\log 1/t)^\alpha}\int_1^\infty \frac{1}{x}\exp(-\tfrac{1}{2}x^2)\,dx,$$

since in the range of integration, $\log 1/x + \frac{1}{2}\log 1/t > \frac{1}{4}\log 1/t$. Thus

$$I_2(t) < \frac{c_7}{t(\log 1/t)^\alpha}$$

for suitable $c_7 > 0$.

$$I_3(t) = \frac{1}{t}\int_{t^{-\frac{1}{4}}}^{\frac{1}{2}t^{-\frac{1}{2}}} \frac{\exp(-\frac{1}{2}x^2)}{x\{\log(1/x)+\frac{1}{2}\log 1/t\}^\alpha}\,dx$$

$$< \frac{c_8}{t}\int_{t^{-\frac{1}{4}}}^\infty \frac{1}{x}\exp(-\tfrac{1}{2}x^2)\,dx,$$

since in the range of integration $\log 1/x + \frac{1}{2}\log 1/t \geqslant \log 2$. Thus

$$I_3(t) < \frac{c_9}{t^{\frac{1}{2}}}$$

for suitable $c_9 > 0$. We have obtained the estimate

$$I(t) < \frac{c_6}{t(\log 1/t)^{\frac{1}{2}\alpha}} + \frac{c_7}{t(\log 1/t)^\alpha} + \frac{c_9}{t^{\frac{1}{2}}} + \lambda_\alpha^{-1}\exp(-\tfrac{1}{4}t) \tag{41}$$

which is valid for $0 < t < \frac{1}{16}$. A simple computation shows that for $t \geqslant \frac{1}{16}$ there is a finite constant c_{10} such that

$$I(t) \leqslant c_{10}. \tag{42}$$

The estimates (41) and (42) show that, if $\alpha > 2$

$$\int_0^1\left\{\int_0^1\left(\int_\Omega \frac{d\omega}{\phi_\alpha(|z(t,\omega)-z(s,\omega)|)}\right)dt\right\}ds < +\infty. \tag{43}$$

By Fubini's theorem, (43) implies that, with probability 1,

$$\int_0^1\left\{\int_0^1 \frac{1}{\phi_\alpha(|z(t,\omega)-z(s,\omega)|)}\,dt\right\}ds < +\infty. \tag{44}$$

But if E is the set $L(0,1;\omega)$ and ν is the measure on E obtained as the image of Lebesgue measure on $[0,1]$, the paths ω which satisfy (44) must give sets $L(0,1;\omega)$ which satisfy (39). Thus for any $\alpha > 2$, there is probability 1 that

$$C^{(\Phi_\alpha)}\{L(0,1;\omega)\} > 0,$$

where

$$\Phi_\alpha(x) = \frac{1}{\phi_\alpha(x)}.$$

By Lemma 8 and (40), there is probability 1 that

$$h_\alpha\text{-}m\{L(0,1;\omega)\} = +\infty. \tag{45}$$

Let α take successively the values

$$\alpha_r = 2+\frac{1}{r} \quad (r = 1, 2, \ldots).$$

Then there is probability 1 that

$$h_\alpha\text{-}m\{L(0,1;\omega)\} = +\infty \quad \text{for} \quad \alpha = \alpha_1, \alpha_2, \ldots.$$

This implies the result stated in the theorem.

REFERENCES

(1) BESICOVITCH, A. S. On existence of subsets of finite measure of sets of infinite measure. *Indag. Math.* 14 (1952), 339–44.

(2) DVORETZKY, A. and ERDŐS, P. Some problems on random walk in space. *Proc. Second Berkeley Symposium* (1951), 353–67.

(3) ERDŐS, P. and TAYLOR, S. J. Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar.* 11 (1960), 137–62.

(4) HAUSDORFF, F. Dimension und äusseres Mass. *Math. Ann.* 79 (1918), 157–79.

(5) KAMETANI, S. On Hausdorff's measures and generalised capacities with some of their applications to the theory of functions. *Jap. J. Math.* 19 (1946), 217–57.

(6) LÉVY, P. *Théorie de l'addition des variables aléatoires* (Paris, 1937).

(7) LÉVY, P. Le mouvement Brownien plan. *Amer. J. Math.* 62 (1940), 487–550.

(8) LÉVY, P. La mesure de Hausdorff de la courbe du mouvement brownien. *Giom. Ist. Ital. Attuari*, 16 (1953), 1–37.

(9) TAYLOR, S. J. The Hausdorff α-dimensional measure of Brownian paths in n-space. *Proc. Camb. Phil. Soc.* 49 (1953), 31–9.

(10) TAYLOR, S. J. On the connection between Hausdorff measures and generalised capacity (to appear shortly).

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