ON CANTOR'S SERIES WITH CONVERGENT $\sum \frac{1}{q_n}$

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Introduction

Let $\{q_n\}$ be an arbitrary sequence of positive integers subjected only to the restriction $q_n \geqq 2$ $(n=1, 2, \ldots)$. Then every real number x $(0 \leqq x < 1)$ can be represented in the form of *Cantor's series*

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{q_1 q_2 \ldots q_n} \tag{1}$$

where the n-th "digit" $\varepsilon_n(x)$ may have the values $0, 1, \ldots, q_n - 1$. The digits $\varepsilon_n(x)$ can be obtained successively starting with $r_0(x) = x$, by the algorithm

$$\varepsilon_n(x) = [q_n r_{n-1}(x)], \quad r_n(x) = (q_n r_{n-1}(x)) \tag{2}$$

where $[t]$ denotes the integral part, and (t) the fractional part of the real number t.

In some previous papers ([1], [2], [3]) the statistical properties of the digits $\varepsilon_n(x)$ valid for almost all x, have been discussed, for the cases when $\sum_{n=1}^{\infty} \frac{1}{q_n}$ is divergent and when it is convergent. (See also [4] and [5]). In the present paper we consider mainly the case when $\sum_{n=1}^{\infty} \frac{1}{q_n}$ is convergent. This case has been considered in [2] from another point of view. The point of view adopted in the present paper is to consider properties of the infinite sequence $\{\varepsilon_n(x)\}$ as a whole; this point of view has led to the formulation and solution of a quite surprising number of questions, which have not been investigated up to now. Most of these questions are interesting only in the case, when $\sum \frac{1}{q_n} < +\infty$; some of them can be raised only under this condition.

Our main tool will be a generalization of the *Borel—Cantelli* lemma, which is proved in § 1. Our results on *Cantor*'s series are contained in §§ 2, 3, 4, and 5.

§ 1. Generalization of the Borel—Cantelli lemma

Let $[X, \mathcal{A}, \mathbf{P}]$ be a probability space in the sense of KOLMOGOROV [6], i. e. X an arbitrary set, whose elements are called "elementary events" and denoted by x, $\mathcal{A}$ a σ-algebra of subsets of X, whose elements are denoted by capital letters (e. g. A, B, etc.), and called events, and $\mathbf{P}(A)$ $(A \in \mathcal{A})$ a probability measure in X and on $\mathcal{A}$. We shall denote by $A+B$ resp. AB the union resp. the intersection of the sets A and B, and by $\bar{A}$ the complementary set of A. We shall denote random variables (i. e. functions defined on X and measurable with respect to $\mathcal{A}$) by greek letters, and denote by $\mathbf{M}(\xi)$ resp. $\mathbf{D}^2(\xi)$ the mean value resp. variance of the random variable $\xi=\xi(x)$. i. e. we put $\mathbf{M}(\xi)=\int\limits_X \xi(x)\,d\mathbf{P}$ and $\mathbf{D}^2(\xi)=\mathbf{M}(\xi^2)-\mathbf{M}^2(\xi)$. If $A_n \subset X$ $(n=1,2,\ldots)$, we denote as usual by $\overline{\lim\limits_{n\to+\infty}} A_n$ the set consisting of those elements x of X which belong to infinitely many A_n, and by $\underline{\lim\limits_{n\to+\infty}} A_n$ the set of those elements x of X which belong to A_n for all $n \geqq n_0(x)$.

The events A and B are called independent if $\mathbf{P}(AB)=\mathbf{P}(A)\mathbf{P}(B)$. A finite or infinite sequence $\{A_n\}$ of events such that any two events of the sequence are independent, is called a sequence of pairwise independent events. If moreover we have $\mathbf{P}(A_{n_1}A_{n_2}\ldots A_{n_r})=\mathbf{P}(A_{n_1})\mathbf{P}(A_{n_2})\ldots\mathbf{P}(A_{n_r})$ for any r-tuple of different events $A_{n_1},\ldots,A_{n_r}$ chosen from the sequence A_n for all $r=2,3,\ldots$, we call the sequence $\{A_n\}$ a sequence of completely independent events.

We shall often use the following well-known

LEMMA A. *If $\{A_n\}$ is an arbitrary sequence of events belonging to a probability space $[X, \mathcal{A}, \mathbf{P}]$ such that $\sum\limits_{n=1}^{\infty}\mathbf{P}(A_n)<+\infty$, then with probability* 1 *only a finite number of the events A_n occur simultaneously, i. e.* $\mathbf{P}(\overline{\lim\limits_{n\to+\infty}} A_n)=0$.

LEMMA A is nothing else as a special case of *Beppo Levi*'s theorem. As a matter of fact, if α_n is a random variable which is equal to 1 if A_n occurs and to 0 if A_n does not occur, then the assertion, that only a finite number of the A_n occur with probability 1 is equivalent with the statement that $\sum\limits_{n=1}^{\infty}\alpha_n$ converges with probability 1 and the condition $\sum\limits_{n=1}^{\infty}\mathbf{P}(A_n)<+\infty$ can be written in the form $\sum\limits_{n=1}^{\infty}\mathbf{M}(\alpha_n)<+\infty$.

The condition $\sum\limits_{n=1}^{\infty}\mathbf{P}(A_n)<+\infty$ of Lemma A is under certain restrictions not only sufficient but also necessary for $\mathbf{P}(\overline{\lim_{n\to+\infty}} A_n)=0$. For example the following result is classical:

LEMMA B. *If $\{A_n\}$ is a sequence of completely independent events and $\sum_{n=1}^{\infty}\mathbf{P}(A_n)=+\infty$, then with probability* 1 *infinitely many among the events A_n occur simultaneously, i. e.* $\mathbf{P}(\overline{\lim_{n\to+\infty}} A_n)=1$.

Lemma A and B together are known under the name: *the lemma of Borel—Cantelli* ([7], [8]).

In this § we shall prove the following generalization of Lemma B.

LEMMA C. *Let $\{A_n\}$ be a sequence of events such that* $\sum_{n=1}^{\infty}\mathbf{P}(A_n)=+\infty$ *and*

$$\lim_{n\to+\infty}\frac{\sum_{k=1}^{n}\sum_{l=1}^{n}\mathbf{P}(A_k A_l)}{\left(\sum_{k=1}^{n}\mathbf{P}(A_k)\right)^2}=1. \tag{1.1}$$

It follows that with probability 1 *infinitely many among the events A_n occur simultaneously, i. e.* $\mathbf{P}(\overline{\lim_{n\to+\infty}} A_n)=1$.

PROOF OF LEMMA C. Let us define α_n as above, i. e. $\alpha_n=1$ or $=0$ according to which the event A_n occurs or not. Then we have $\mathbf{M}(\alpha_k)=\mathbf{P}(A_k)$ and $\mathbf{M}(\alpha_k\alpha_l)=\mathbf{P}(A_k A_l)$ and thus putting $\eta_n=\sum_{k=1}^{n}\alpha_k$ we have

$$\frac{\sum_{k=1}^{n}\sum_{l=1}^{n}\mathbf{P}(A_k A_l)}{\left(\sum_{k=1}^{n}\mathbf{P}(A_k)\right)^2}=\frac{\mathbf{M}(\eta_n^2)}{\mathbf{M}^2(\eta_n)}$$

Thus condition (1. 1) can be written in the equivalent form

$$\lim_{n\to+\infty}\frac{\mathbf{M}(\eta_n^2)}{\mathbf{M}^2(\eta_n)}=1 \tag{1.2}$$

or as $\mathbf{M}(\eta_n^2)=\mathbf{D}^2(\eta_n)+\mathbf{M}^2(\eta_n)$, also in the form

$$\lim_{n\to+\infty}\frac{\mathbf{D}^2(\eta_n)}{\mathbf{M}^2(\eta_n)}=0. \tag{1.3}$$

Now by the inequality of *Chebyshev* according to which for any random variable η we have

$$\mathbf{P}(|\eta-\mathbf{M}(\eta)|\geqq\lambda\mathbf{D}(\eta))\leqq\frac{1}{\lambda^2}\quad\text{if}\quad\lambda>1, \tag{1.4}$$

we have for any ε with $0<\varepsilon<1$

$$\mathbf{P}(\eta_n\leqq(1-\varepsilon)\mathbf{M}(\eta_n))\leqq\frac{\mathbf{D}^2(\eta_n)}{\varepsilon^2\mathbf{M}^2(\eta_n)}. \tag{1.5}$$

If (1. 3) holds, we can find a sequence n_k $(n_1<n_2<\cdots)$ such that

$$\sum_{k=1}^{\infty}\frac{\mathbf{D}^2(\eta_{n_k})}{\mathbf{M}^2(\eta_{n_k})}<+\infty. \tag{1.6}$$

It follows from (1. 5) and (1. 6) that

$$\sum_{k=1}^{\infty}\mathbf{P}(\eta_{n_k}\leqq(1-\varepsilon)\,\mathbf{M}(\eta_{n_k}))<+\infty. \tag{1.7}$$

Using Lemma A it follows that with probability 1 $\eta_{n_k}\geqq(1-\varepsilon)\mathbf{M}(\eta_{n_k})$ except for a finite number of values of k. As by supposition $\lim_{k\to+\infty}\mathbf{M}(\eta_{n_k})=+\infty$, it follows that η_{n_k} tends to $+\infty$ with probability 1, which implies that $\mathbf{P}(\overline{\lim_{n\to+\infty}}A_n)=1$, what was to be proved.

REMARK. Clearly the condition (1. 1) is satisfied if the events A_n are pairwise independent and $\sum_{n=1}^{\infty}\mathbf{P}(A_n)=+\infty$, because in this case

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\mathbf{P}(A_kA_l)=\left(\sum_{k=1}^{n}\mathbf{P}(A_k)\right)^2+\sum_{k=1}^{n}\mathbf{P}(A_k)\,(1-\mathbf{P}(A_k)) \tag{1.8}$$

for all n. Thus condition (1. 1) can be regarded as a condition ensuring that the events A_n should be in a certain sense pairwise weakly dependent and Lemma C contains as a particular case the following

COROLLARY 1. If the events A_n are *pairwise* independent, and $\sum\mathbf{P}(A_n)=+\infty$, then with probability 1 infinitely many of the events A_n occur simultaneously.

COROLLARY 2. If $\mathbf{P}(A_kA_l)\leqq\mathbf{P}(A_k)\mathbf{P}(A_l)$ for $k\neq l$ (i. e. if the events A_n are pairwise negatively correlated) and $\sum_{n=1}^{\infty}\mathbf{P}(A_n)=+\infty$ then with probability 1 infinitely many of the events A_n occur simultaneously.

PROOF OF COROLLARY 2. If $\mathbf{P}(A_kA_l)\leqq\mathbf{P}(A_k)\mathbf{P}A_l)$ for $k\neq l$ we have

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\mathbf{P}(A_kA_l)\leqq\left(\sum_{k=1}^{n}\mathbf{P}(A_k)\right)^2+\sum_{k=1}^{n}\mathbf{P}(A_k)(1-\mathbf{P}(A_k)) \tag{1.9}$$

thus condition (1. 1) is satisfied provided that the series $\sum_{n=1}^{\infty}\mathbf{P}(A_n)$ is divergent.

§ 2. On the frequency of the digits in Cantor's series

Let us consider the probability space $[X, \mathcal{A}, \mathbf{P}]$ where X is the interval $[0, 1)$, $\mathcal{A}$ the family of Lebesgue measurable subsets of X and $\mathbf{P}(A)$ the ordinary Lebesgue measure of $A \in \mathcal{A}$. Thus the Lebesgue measure of a measurable subset A of the interval $[0, 1)$ is interpreted as the probability of a random point falling into A. With this interpretation the digits $\varepsilon_n(x)$ as well as any other measurable functions $f(x)$ of x will be considered as random variables. Clearly we have

$$\mathbf{P}(\varepsilon_n(x) = k) = \frac{1}{q_n} \quad \text{for} \quad k = 0, 1, \ldots, q_n - 1, \tag{2.1}$$

further if $n_1 < n_2 < \cdots < n_r$ $(r = 1, 2, \ldots)$

$$\mathbf{P}(\varepsilon_{n_1}(x) = k_1, \ldots, \varepsilon_{n_r}(x) = k_r) = \frac{1}{q_{n_1} q_{n_2} \ldots q_{n_r}} \tag{2.2}$$
$$\text{if} \quad 0 \leqq k_j \leqq q_{n_j} - 1 \quad \text{for} \quad j = 1, \ldots, r.$$

(2.2) expresses the fact, that the random variables $\varepsilon_n(x)$ are completely independent.

Let us suppose from now on that

$$\sum_{n=1}^{\infty} \frac{1}{q_n} < +\infty \tag{2.3}$$

except when the contrary is explicitly stated.

By (2.2) and (2.3) it follows that for any $k = 0, 1, \ldots$ we have

$$\sum_{n=1}^{\infty} \mathbf{P}(\varepsilon_n(x) = k) < +\infty. \tag{2.4}$$

Moreover it follows from (2.3) that for any positive integer N

$$\sum_{n=1}^{\infty} \mathbf{P}(\varepsilon_n(x) < N) = \sum_{q_n \leqq N} 1 + \sum_{N < q_n} \frac{N}{q_n} < +\infty. \tag{2.5}$$

Thus *the sequence* $\varepsilon_n(x)$ *tends to* $+\infty$ *for almost all* x. As a matter of fact, by Lemma A for almost all x and for any N $\varepsilon_n(x) < N$ only for a finite number of values of n, which is equivalent with the assertion that $\lim\limits_{n \to +\infty} \varepsilon_n(x) = +\infty$ for almost all x.

By Lemma A it follows from (2.4) that for almost all x each number k occurs only a finite number of times in the sequence $\varepsilon_n(x)$; thus if we denote by $\nu_{k,n}(x)$ $(k = 0, 1, \ldots;\ n = 1, 2, \ldots)$ the number of occurrences of the number k in the sequence $\varepsilon_n(x), \varepsilon_{n+1}(x), \ldots$ then $\nu_{k,n}(x)$ is an almost everywhere finite and measurable function, i. e. a well defined random variable. We shall write for the sake of simplicity $\nu_{k,1}(x) = \nu_k(x)$.

It is quite easy to determine the probability distribution of $\nu_{k,n}(x)$. Putting

$$\mathrm{P}_{k,n}(s) = \mathbf{P}(\nu_{k,n}(x) = s) \tag{2.6}$$

we have evidently by (2. 2)

$$\mathrm{P}_{k,n}(s) = \sum_{\substack{n \le n_1 < n_2 < \cdots < n_s \\ q_{n_r} > k \ (r=1,2,\ldots,s)}} \frac{1}{q_{n_1} q_{n_2} \ldots q_{n_s}} \prod_{\substack{j \ne n_r, \, 1 \le r \le s) \\ q_j > k \\ j \ge n}} \left(1 - \frac{1}{q_j}\right). \tag{2.7}$$

It follows from (2. 7) that

$$\mathrm{P}_{k,n}(s) = \prod_{\substack{q_j > k \\ j \ge n}} \left(1 - \frac{1}{q_j}\right) \left(\sum_{\substack{n \le n_1 < n_2 < \cdots < n_s \\ q_{n_r} > k \ (r=1,2,\ldots,s)}} \frac{1}{(q_{n_1} - 1) \ldots (q_{n_s} - 1)} \right) \tag{2.8}$$

and thus we obtain for the generating function of the random variable $\nu_{k,n}$ the simple formula

$$\sum_{s=0}^{\infty} \mathrm{P}_{k,n}(s) z^s = \prod_{\substack{q_j > k \\ j \ge n}} \left(1 + \frac{z-1}{q_j}\right). \tag{2.9}$$

(The special case $n = 1$ of formula (2. 9) is given *already in* [2].) *Clearly*

$$\mathbf{M}(\nu_{k,n}(x)) = \sum_{j=n}^{\infty} \mathbf{P}(\varepsilon_j(x) = k) = \sum_{\substack{q_j > k \\ j \ge n}} \frac{1}{q_j} < +\infty. \tag{2.10}$$

Thus the mean value of the occurrence of each digit k $(k = 0, 1, \ldots)$ is finite. Now *let us put*

$$m_n(x) = \sup_{(k)} \nu_{k,n}(x) \tag{2.11}$$

and

$$m(x) = \lim_{n \to +\infty} m_n(x). \tag{2.12}$$

(As $m_n(x) \ge m_{n+1}(x) \ge 0$ the limit (2. 12) always exists.) $m_n(x)$ and $m(x)$ are generalized random variables *in the sense that they may take on the value* $+\infty$ *on a set of positive* measure. Clearly $m(x)$ is a Baire-function of the independent random variables $\varepsilon_n(x)$ $(n = 1, 2, \ldots)$ which does not change its value if a finite number of the $\varepsilon_n(x)$ change their value. Thus, according to the law of 0 or 1 (see [6]) the probability $\mathbf{P}(m(x) = s)$ is for any $s = 1, 2, \ldots$ either 0 or 1. Similarly *the probability* $\mathbf{P}(m(x) = +\infty)$ is either 0 or 1.

Our first result decides when these two possibilities occur.

THEOREM 1. *Let us suppose that* $q_n \leqq q_{n+1}$ *and* $\sum_{n=1}^{\infty} \frac{1}{q_n} < +\infty$ *and put*

$$R_n = \sum_{j=n}^{\infty} \frac{1}{q_j} \qquad (n = 1, 2, \ldots) \tag{2.13}$$

If $\sum_{n=1}^{\infty} R_n^{s-1} = +\infty$ *but* $\sum_{n=1}^{\infty} R_n^s < +\infty$ *for some positive integer* s, *then we have*

$$\mathbf{P}(m(x) = s) = 1. \tag{2.14}$$

We have

$$\mathbf{P}(m(x) = +\infty) = 1 \tag{2.15}$$

if and only if $\sum_{n=1}^{\infty} R_n^s = +\infty$ *for all* $s = 1, 2, \ldots$.

REMARK 1. First of all, the assumption that $q_n \geqq q_{n+1}$ does not restrict the generality, as clearly this condition can be fulfilled always by reordening the q_n according to their size, and this reordening, though affects the expansion (1), does not affect the joint distribution of the random variables $\varepsilon_n(x)$ and thus does not influence such properties of the sequence $\varepsilon_n(x)$ which depend only on the values and not on the arrangement of these variables. Especially such a reordening does not affect the distribution of the variable $m(x)$, because $m(x) = s$ means that there can be found an infinity of s-tuples of different positive integers $n_1, n_2, \ldots, n_s$ such that $\varepsilon_{n_1}(x) = \varepsilon_{n_2}(x) = \cdots = \varepsilon_{n_s}(x)$ but only a finite number of $s+1$-tuples $m_1, m_2, \ldots, m_{s+1}$ such that $\varepsilon_{m_1}(x) = \varepsilon_{m_2}(x) = \cdots = \varepsilon_{m_{s+1}}(x)$.

REMARK 2. Let us put $\mu(x) = \overline{\lim}_{k \to +\infty} \nu_k(x)$. It is easy to see that $\mathbf{P}(m(x) = \mu(x)) = 1$. As a matter of fact, if $m(x) \geqq s$, there are an infinity of s-tuplet, $n_1, \ldots, n_s$ such that $\varepsilon_{n_1}(x) = \varepsilon_{n_2}(x) = \cdots = \varepsilon_{n_s}(x)$; as we have $\lim_{n \to +\infty} \varepsilon_n(x) = +\infty$ for almost all x, this means that $\mu(x) \geqq s$. Conversely, if $\mu(x) \geqq s$ then there are an infinity of s-tuples of equal digits, and so $m(x) \geqq s$. Thus the assertions of Theorem 1 hold for $\mu(x)$ instead of $m(x)$ too.

PROOF OF THEOREM 1. Clearly to show that

$$\sum_{n=1}^{\infty} R_n^s < +\infty \tag{2.16}$$

implies $m(x) \leqq s$ for almost all x, it suffices to prove that the series

$$\sum_{1 \leqq n_1 < n_2 < \cdots < n_{s+1}} \mathbf{P}(\varepsilon_{n_1}(x) = \varepsilon_{n_2}(x) = \cdots = \varepsilon_{n_{s+1}}(x)) \tag{2.17}$$

converges. As a matter of fact, if the series (2.17) converges, then by

Lemma A for almost all x only a finite number of the events $\varepsilon_{n_1}(x) = \cdots = \varepsilon_{n_{s+1}}(x)$ will occur, which implies $m(x) \leqq s$. But if $n_1 < n_2 < \cdots < n_{s+1}$, then

$$\mathbf{P}(\varepsilon_{n_1}(x) = \varepsilon_{n_2}(x) = \cdots = \varepsilon_{n_{s+1}}(x)) = \frac{1}{q_{n_2} q_{n_3} \ldots q_{n_{s+1}}} \tag{2.18}$$

and thus the series (2. 17) is equal to the series

$$\sum_{1 < n_2 < n_3 < \cdots < n_{s+1}} \frac{n_2 - 1}{q_{n_2} \ldots q_{n_{s+1}}}. \tag{2.19}$$

Now we have clearly

$$\sum_{1 < n_2 < n_3 < \cdots < n_{s+1}} \frac{n_2 - 1}{q_{n_2} \ldots q_{n_{s+1}}} \leqq \frac{1}{s!} \sum_{n=1}^{\infty} R_n^s. \tag{2.20}$$

Thus if (2.16) holds, then the series (2. 17) converges, which proves our assertion, that (2. 16) implies $m(x) \leqq s$ for almost all x. Let us suppose now that

$$\sum_{n=1}^{\infty} R_n^{s-1} = +\infty. \tag{2.21}$$

Let us denote by $A_{n_1 n_2 \ldots n_s}$ the event $\varepsilon_{n_1}(x) = \varepsilon_{n_2}(x) = \cdots = \varepsilon_{n_s}(x)$ $(1 \leqq n_1 < n_2 < \cdots < n_s)$. Then as above, it follows that

$$\sum_{1 \leqq n_1 < n_2 < \cdots < n_s} \mathbf{P}(A_{n_1 n_2 \ldots n_s}) = \sum_{n=2}^{\infty} \sum_{n \leqq n_2 < n_3 < \cdots < n_s} \frac{1}{q_{n_2} \ldots q_{n_s}}. \tag{2.22}$$

Now we use the inequality

$$\sum_{1 \leqq i_1 < i_2 < \cdots < i_k \leqq N} a_{i_1} a_{i_2} \ldots a_{i_k} \geqq \frac{1}{k!} \left(\sum_{i=1}^{N} a_i \right)^k \left(1 - \binom{k}{2} \frac{\sum_{i=1}^{N} a_i^2}{\left(\sum_{i=1}^{N} a_i \right)^2} \right) \tag{2.23}$$

valid for any sequence a_i of positive numbers and for $k = 1, 2, \ldots$. (2. 23) is trivial for $k = 1$ and $k = 2$ and follows for arbitrary k easily by induction. It follows that

$$\sum_{n \leqq n_2 < n_3 < \cdots < n_s} \frac{1}{q_{n_2} \ldots q_{n_s}} \geqq \frac{R_n^{s-1}}{(s-1)!} \qquad \text{if} \quad s = 2 \tag{2.24a}$$

and

$$\sum_{n \leqq n_2 < n_3 < \cdots < n_s} \frac{1}{q_{n_2} q_{n_3} \ldots q_{n_s}} \geqq \frac{R_n^{s-1}}{(s-1)!} \left(1 - \binom{s-1}{2} \frac{\sum_{j=n}^{\infty} \frac{1}{q_j^2}}{R_n^2} \right) \quad \text{if} \quad s \geqq 3. \tag{2.24b}$$

As evidently

$$\sum_{n=1}^{\infty} R_n^{s-3} \cdot \sum_{j=n}^{\infty} \frac{1}{q_j^2} \leqq R_1^{s-3} \sum_{j=1}^{\infty} \frac{j}{q_j^2}$$

and the series $\sum_{j=1}^{\infty} \frac{j}{q_j^2}$ is convergent, because

$$\sum_{j=1}^{\infty} \frac{j}{q_j^2} = \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \frac{1}{q_j^2} \leqq \sum_{n=1}^{\infty} \frac{1}{q_n} \sum_{j=n}^{\infty} \frac{1}{q_j} \leqq \left(\sum_{n=1}^{\infty} \frac{1}{q_n} \right)^2$$

it follows from (2. 21), (2. 22) and (2. 24a) resp. (2. 24b) that

$$\sum_{1 \leqq n_1 < n_2 < \dots < n_s} \mathbf{P}(A_{n_1 n_2 \dots n_s}) = +\infty. \tag{2. 25}$$

We shall apply now Lemma C For this purpose we have to verify the fulfillment of condition (1. 1).

Let us arrange the s-tuples of positive integers $n_1 < n_2 < \dots < n_s$ in lexicographic order. We have evidently, putting

$$B_N^{(s)} = \sum_{n_1 < n_2 < \dots < n_s \leqq N} \mathbf{P}(A_{n_1 n_2 \dots n_s}), \tag{2. 26}$$

$$\sum_{\substack{n_1 < n_2 < \dots < n_s \leqq N \\ m_1 < m_2 < \dots < m_s \leqq N}} \mathbf{P}(A_{n_1 \dots n_s} A_{m_1 \dots m_s}) \leqq (B_N^{(s)})^2 + \sum_{k=1}^{s} \binom{s}{k} \binom{2s-k}{s} B_N^{(2s-k)}. \tag{2. 27}$$

Thus we have

$$\frac{\sum_{\substack{n_1 < \dots < n_s \leqq N \\ m_1 < \dots < m_s \leqq N}} \mathbf{P}(A_{n_1 \dots n_s} A_{m_1 \dots m_s})}{\left(\sum_{n_1 < \dots < n_s \leqq N} \mathbf{P}(A_{n_1 \dots} A_{n_s}) \right)^2} \leqq 1 + \frac{\sum_{k=1}^{s} \binom{s}{k} \binom{2s-k}{s} \frac{R_1^{2s-k}}{(2s-k)!}}{B_N^{(s)}} \tag{2. 28}$$

which shows, that condition (1. 1) is satisfied, because by supposition $\lim_{N \to +\infty} B_N^{(s)} = +\infty$.

Thus we may apply Lemma C and it follows, that with probability 1 an infinity of the events $A_{n_1 \dots n_s}$ occur simultaneously. But this means that $\mathbf{P}(m(x) \geqq s) = 1$. Thus if (2. 16) and (2. 21) both hold, we have $\mathbf{P}(m(x) \leqq s) = \mathbf{P}(m(x) \geqq s) = 1$ and thus $\mathbf{P}(m(x) = s) = 1$.

On the other hand if (2. 21) holds, for $s = 2, 3, \dots$ then $\mathbf{P}(m(x) \geqq s) = 1$ for $s = 2, 3, \dots$ and thus $\mathbf{P}(m(x) = +\infty) = 1$.

An other question, related with Theorem 1 is the following: how many of the first N digits $\varepsilon_1(x), \dots, \varepsilon_N(x)$ are different? If we denote this number by $D_N(x)$ and by $C_{N,k}(x)$ the number of equal k-tuples among the first N digits, we have clearly

$$N - C_{N,2}(x) \leqq D_N(x) \leqq N. \tag{2. 29}$$

It follows by what has been proved above that $\frac{D_N(x)}{N}$ tends stochastically to 1.

By a somewhat more refined argument it can be proved that $\frac{D_N(x)}{N}$ tends almost everywhere to 1, i. e. the following theorem is valid:

THEOREM 2. *Suppose* $\sum_{n=1}^{\infty} \frac{1}{q_n} < +\infty$. *Let* $D_N(x)$ *denote the number of different numbers in the sequence* $\varepsilon_1(x), \ldots, \varepsilon_N(x)$. *Then for almost every* x *we have*

$$\lim_{N\to+\infty} \frac{D_N(x)}{N} = 1. \tag{2. 30}$$

PROOF. With regards to (2. 29) to prove Theorem 2 it suffices to show that

$$\lim_{N\to+\infty} \frac{C_{N,2}(x)}{N} = 0 \tag{2. 31}$$

for almost every x. Now we have $\mathbf{M}(C_{N,2}(x)) = \sum_{n=1}^{N} \frac{n}{q_n} = Nh_N$ where $\lim_{N\to+\infty} h_N = 0$ further $\mathbf{D}^2(C_{N,2}(x)) \leq KNh_N$ where K is a constant. It follows by the inequality of *Chebyshev* that if $\varepsilon > 0$ and N is so large that $h_N < \varepsilon/2$, we have

$$\mathbf{P}(C_{N,2}(x) > \varepsilon N) < \frac{2Kh_N}{\varepsilon N}. \tag{2. 32}$$

It follows that

$$\sum_{n=1}^{\infty} \mathbf{P}(C_{n^2,2}(x) > \varepsilon n^2) < +\infty. \tag{2. 33}$$

It follows by Lemma A that

$$\lim_{n\to+\infty} \frac{C_{n^2,2}(x)}{n^2} = 0 \tag{2. 34}$$

for almost every x, and therefore by (2. 29)

$$\lim_{n\to+\infty} \frac{D_{n^2}(x)}{n^2} = 1 \tag{2. 35}$$

for almost every x. But clearly if $n^2 < N < (n+1)^2$ we have

$$\frac{D_{n^2}(x)}{n^2} \cdot \left(\frac{n}{n+1}\right)^2 \leq \frac{D_N(x)}{N} \leq 1 \tag{2. 36}$$

and thus it follows that (2. 30) holds for almost all x. This proves Theorem 2.

REMARK. For the validity of Theorem 2 it is sufficient — as can be seen from the above proof — to suppose instead of the convergence of $\sum_{n=1}^{\infty} \frac{1}{q_n}$ only that $\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{n}{q_n} = 0$.

§ 3. Some other statistical properties of the digits

It seems plausible that if q_n tends very rapidly to $+\infty$ the sequence $\varepsilon_n(x)$ of digits will be increasing from some point onwards. This is in fact true, as is shown by the following

THEOREM 3. *The necessary and sufficient condition for the sequence $\varepsilon_n(x)$ to be increasing for $n \geqq n_0(x)$ for almost all x is that the condition*

$$\sum_{n=1}^{\infty} \frac{q_n}{q_{n+1}} < +\infty \tag{3.1}$$

should hold.

PROOF. Clearly

$$\mathbf{P}(\varepsilon_{n+1}(x) \leqq \varepsilon_n(x)) = \sum_{j=0}^{q_n-1} \frac{q_n - j}{q_n q_{n+1}} = \frac{q_n+1}{2q_{n+1}}. \tag{3.2}$$

Thus if (3. 1) holds, then

$$\sum_{n=1}^{\infty} \mathbf{P}(\varepsilon_{n+1}(x) \leqq \varepsilon_n(x)) < +\infty \tag{3.3}$$

and therefore by Lemma A for almost all x, $\varepsilon_{n+1}(x) > \varepsilon_n(x)$ except for a finite number of values of n. This proves the first part of Theorem 3.

As regards the second part, let us suppose

$$\sum_{n=1}^{\infty} \frac{q_n}{q_{n+1}} = +\infty. \tag{3.4}$$

In this case

$$\sum_{n=1}^{N} \sum_{m=1}^{N} \mathbf{P}(\varepsilon_{n+1}(x) \leqq \varepsilon_n(x), \varepsilon_{m+1}(x) \leqq \varepsilon_m(x)) \leqq \left(\sum_{n=1}^{N-1} \mathbf{P}(\varepsilon_{n+1}(x) \leqq \varepsilon_n(x)\right)^2 + \\ + 2\sum_{n=1}^{N-2} \mathbf{P}(\varepsilon_{n+2}(x) \leqq \varepsilon_{n+1}(x) \leqq \varepsilon_n(x)). \tag{3.5}$$

As

$$\mathbf{P}(\varepsilon_{n+2}(x) \leqq \varepsilon_{n+1}(x) \leqq \varepsilon_n(x)) = \sum_{j=0}^{q_n-1} \sum_{k=j}^{q_n-1} \frac{q_n - k}{q_n q_{n+1} q_{n+2}} \leqq \frac{q_n}{3q_{n+1}} \tag{3.6}$$

it follows that condition (1. 1) of Lemma C is fulfilled. This implies that for almost all x $\varepsilon_{n+1}(x) \leqq \varepsilon_n(x)$ for an infinity of values of n; thus Theorem 3 is proved.

We have seen, that $\varepsilon_n(x)$ tends for almost all x to $+\infty$. One may ask what can be said about the speed with which $\varepsilon_n(x)$ increases. In this direction one can easily prove results of the following type:

THEOREM 4. $\sum_{n=1}^{\infty} \frac{1}{1+\varepsilon_n(x)} < +\infty$ *for almost all x if and only if* $\sum_{n=1}^{\infty} \frac{\log q_n}{q_n} < +\infty$.

PROOF OF THEOREM 4. The proof of the sufficiency is immediate by the theorem of *B. Levi*, taking into account that

$$\mathbf{M}\left(\frac{1}{1+\varepsilon_n(x)}\right) = \frac{1}{q_n} \sum_{k=1}^{q_n} \frac{1}{k} \tag{3. 7}$$

As the variables $\varepsilon_n(x)$ are completely independent, the necessity follows from the three-series theorem of *Kolmogorov* [6].

§ 4. On the set of all digits

In this § we consider the following question: what can be said about the set $S(x)$ of those positive integers, which occur at least once in the sequence $\{\varepsilon_n(x)\}$. Clearly the probability that a given number k is not contained in $S(x)$ is equal to $\prod_{k<q_n} \left(1-\frac{1}{q_n}\right)$ and is thus positive for all k. Moreover, it is not difficult to find an infinite sequence of integers k_j $(j=1, 2, \ldots)$ such that with probability 1 only a finite number of elements of the sequence k_j are contained in the sequence $\varepsilon_n(x)$. As a matter of fact

$$\mathbf{P}(k \in S(x)) = 1 - \prod_{k<q_n} \left(1-\frac{1}{q_n}\right) \tag{4. 1}$$

and thus

$$\lim_{k\to+\infty} \mathbf{P}(k \in S(x)) = 0. \tag{4. 2}$$

Therefore an infinite sequence $k_1 < k_2 < \cdots < k_j < \cdots$ can be found (depending of course on the sequence q_n) such that

$$\sum_{j=1}^{\infty} \mathbf{P}(k_j \in S(x)) < +\infty. \tag{4. 3}$$

By Lemma A our assertion follows.

Clearly we have also by the general formula

$$\mathbf{P}(AB) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A+B) \tag{4. 4}$$

and by (4. 1) if $k<j$

$$\mathbf{P}(k \in S(x), j \in S(x)) = 1 + \prod_{k<q_n\leq j} \left(1-\frac{1}{q_n}\right) \prod_{j<q_n} \left(1-\frac{2}{q_n}\right) - \\ - \prod_{k<q_n} \left(1-\frac{1}{q_n}\right) - \prod_{j<q_n} \left(1-\frac{1}{q_n}\right). \tag{4. 5}$$

As $\left(1-\frac{2}{q_n}\right) \leqq \left(1-\frac{1}{q_n}\right)^2$

$$\mathbf{P}(k \in S(x), j \in S(x)) \leqq \mathbf{P}(k \in S(x))\mathbf{P}(j \in S(x)) \tag{4.6}$$

if $j \neq k$, and therefore if $\{k_j\}$ is such a sequence that (4.3) holds then by Corollary 2 of Lemma C with probability 1 $S(x)$ contains an infinity of elements of the sequence $\{k_j\}$. Clearly if k is sufficiently large so as to ensure

$$\sum_{q_n > k} \frac{1}{q_n} < \frac{1}{2} \tag{4.7}$$

we have

$$\sum_{q_n > k} \frac{1}{q_n} \geqq 1 - \prod_{q_n > k}\left(1-\frac{1}{q_n}\right) \geqq \frac{1}{2} \sum_{q_n > k} \frac{1}{q_n} \tag{4.8}$$

and thus putting

$$K(x) = \sum_{k_j < x} 1 \tag{4.9}$$

with respect to (4.1) and (4.8) the series (4.3) is convergent or divergent according to whether the series

$$\sum_{j=1}^{\infty} \sum_{k_j < q_n} \frac{1}{q_n} = \sum_{n=1}^{\infty} \frac{K(q_n)}{q_n} \tag{4.10}$$

is convergent or divergent.

Thus we have proved the following

THEOREM 5. *Let $k_1 < k_2 < \cdots < k_j < \cdots$ be an arbitrary infinite sequence of positive integers and define $K(x)$ by (4.9). The set $S(x)$ of all positive integers occurring at least once in the sequence $\{\varepsilon_n(x)\}$ contains for almost all x either a finite or an infinite number of elements of the sequence k_j according to whether the series*

$$\sum_{n=1}^{\infty} \frac{K(q_n)}{q_n} \tag{4.11}$$

converges or diverges.

EXAMPLE. If $q_n = n^2$, then $S(x)$ contains for almost all x only a finite number of elements of the sequence $k_j = j^3$, but an infinite number of elements of the sequence $k_j = j^2$.

It follows easily from Theorem 5 that if the sequence $\{k_j\}$ has positive lower density, i. e. if

$$\varliminf_{x \to +\infty} \frac{K(x)}{x} = \alpha > 0 \tag{4.12}$$

then $S(x)$ contains with probability 1 an infinite number of elements of the sequence $\{k_j\}$, because in this case $\frac{K(q_n)}{q_n}$ does not tend to 0, and thus the series (4. 11) is divergent. If q_n does not increase too rapidly, for instance if $\frac{q_{n+1}}{q_n} \leqq C$ where $C > 0$ is a constant, then the same holds also under the weaker assumption that

$$\overline{\lim_{x\to+\infty}} \frac{K(x)}{x} = \beta > 0 \tag{4.13}$$

i. e. that the sequence $\{k_j\}$ has positive upper density, because in this case if $q_{n-1} \leqq x < q_n$ then

$$\frac{K(q_n)}{q_n} \geqq \frac{K(x)}{q_n} \geqq \frac{1}{C}\,\frac{K(x)}{x}$$

and thus (4. 13) implies $\overline{\lim_{n\to+\infty}} \frac{K(q_n)}{q_n} \geqq \frac{\beta}{A} > 0$ and thus the divergence of the series (4. 11). If however $q_n = 2^{2^n}$ and $\{k_j\}$ consists of the numbers $2^{2^n}+1, \ldots, 2^{2^{n}+1}$ then the upper density of the sequence $\{k_j\}$ is 1/2 but (4. 11) is convergent.

Now we prove the following

THEOREM 6. *The density of* $S(x)$ *is with probability* 1 *equal to* 0.

PROOF. Let $\alpha_N(x)$ denote the number of those $\varepsilon_n(x)$ $(n = 1, 2, \ldots)$ which are $\leqq N$. Clearly if we prove that

$$\mathbf{P}\left(\lim_{N\to+\infty} \frac{\alpha_N(x)}{N} = 0\right) = 1 \tag{4.14}$$

then the assertion of Theorem 6 follows. To prove (4. 14), by Lemma A is sufficient to show that the series

$$\sum_{k=1}^{\infty} \mathbf{P}\left(\frac{\alpha_{2^k}(x)}{2^k} \geqq \varepsilon\right) \tag{4.15}$$

is convergent for any $\varepsilon > 0$. As a matter of fact the convergence of the series (4. 15) implies that for almost all x

$$\lim_{k\to+\infty} \frac{\alpha_{2^k}(x)}{2^k} = 0 \tag{4.16}$$

and as for $2^k \leqq N < 2^{k+1}$, we have $\frac{\alpha_N(x)}{N} \leqq 2 \cdot \frac{\alpha_{2^{k+1}}(x)}{2^{k+1}}$ it follows that

$$\lim_{N\to+\infty} \frac{\alpha_N(x)}{N} = 0 \tag{4.17}$$

for almost all x. As

$$\mathbf{M}(\alpha_N(x)) = \sum_{q_n \leqq N} 1 + \sum_{N < q_n} \frac{N}{q_n} = N d_N \tag{4.18}$$

where $\lim_{N \to +\infty} d_N = 0$ and

$$\mathbf{D}^2(\alpha_N(x)) = \sum_{q_n > N} \frac{N}{q_n}\left(1 - \frac{N}{q_n}\right) \leqq N d_N \tag{4.19}$$

it follows by the inequality of *Chebyshev* that if N is so large that $d_N < \varepsilon/2$, then

$$\mathbf{P}(\alpha_N(x) \geqq N\varepsilon) \leqq \frac{4 d_N}{N\varepsilon^2} < \frac{2}{N\varepsilon}. \tag{4.20}$$

It follows that the series (4. 15) converges, which, as has been pointed out above, proves Theorem 6.

§ 5. On the order of magnitude of $\nu_k(x)$

We denote again by $\nu_k(x)$ the number of occurrences of the number k $(k = 0, 1, \ldots)$ in the sequence $\{\varepsilon_n(x)\}$.

In this § we prove

THEOREM 7. *Let $\{q_n\}$ be an arbitrary sequence of integers $(q_n \geqq 2)$ for which $\sum_{n=1}^{\infty} \frac{1}{q_n} < +\infty$. If C is an arbitrary positive number, then for almost all x*

$$\nu_k(x) \geqq \frac{\log k}{\log\log k} + \frac{\log k \cdot \log\log\log k}{(\log\log k)^2} - C\frac{\log k}{(\log\log k)^2} \tag{5.1}$$

holds at most for a finite number of values of k.

REMARK. It is remarkable, that the growth of $\nu_k(x)$ depends only so weakly on the order of magnitude of q_n, that such an estimate as furnished by Theorem 7 can be given for all sequences q_n. The result of Theorem 7 is best possible as is shown by

THEOREM 8. *If $g(k)$ is an arbitrary sequence of numbers tending to $+\infty$, one can choose the sequence $\{q_n\}$ so that $\sum_{n=1}^{\infty} \frac{1}{q_n} < +\infty$ and*

$$\nu_k(x) \geqq \frac{\log k}{\log\log k} + \frac{\log k \cdot \log\log\log k}{(\log\log k)^2} - g(k)\frac{\log k}{(\log\log k)^2} \tag{5.2}$$

is satisfied for almost all x for an infinity of values of k.

Proof of Theorem 7. We have by (2. 7) for $N \geqq 1$

$$\mathbf{P}(\nu_k(x) \geqq N) = \sum_{s=N}^{\infty} \sum_{\substack{n_1<n_2<\dots<n_s \\ q_{n_r}>k\ (r=1,2,\dots,s)}} \frac{1}{q_{n_1} q_{n_2} \dots q_{n_s}} \prod_{\substack{j\neq n_r\ (1\leqq r\leqq s) \\ q_j>k}} \left(1-\frac{1}{q_j}\right) \tag{5. 3}$$

and thus putting

$$r_k = \sum_{q_n>k} \frac{1}{q_n} \tag{5. 4}$$

we have

$$\mathbf{P}(\nu_k(x) \geqq N) \geqq \sum_{s=N}^{\infty} \frac{r_k^s}{S!}. \tag{5. 5}$$

Let $d>0$ be an arbitrary positive number, and choose k_d so large that for $k \geqq k_d$ we should have $r_k \leqq e^{-d}$; then we obtain for $k \geqq k_d$

$$\mathbf{P}(\nu_k(x) \geqq N) \leqq \frac{2e^{-Nd}}{N!}. \tag{5. 6}$$

Thus if

$$N(k) = \frac{\log k}{\log\log k} + \frac{\log k \cdot \log\log\log k}{(\log\log k)^2} - \frac{C \log k}{(\log\log k)^2} \tag{5. 7}$$

we have

$$\sum_{k=k_d}^{\infty} \mathbf{P}(\nu_k(x) \geqq N(k)) \leqq 2 \sum_{k=k_d}^{\infty} \frac{e^{-dN(k)}}{N(k)!}. \tag{5. 8}$$

As by Stirling's formula

$$\log N(k)! = \log k - \frac{(C+1)\log k}{\log\log k} + O\left(\frac{\log k\,(\log\log\log k)^2}{(\log\log k)^2}\right) \tag{5. 9}$$

it follows

$$\mathbf{P}(\nu_k(x) \geqq N(k)) \leqq \frac{e^{(C+1-d)\frac{\log k}{\log\log k}+O\left(\frac{\log k\,(\log\log\log k)^2}{(\log\log k)^2}\right)}}{k}. \tag{5. 10}$$

It follows by choosing $d > C+1$ that the series (5. 8) converges. Thus we may apply Lemma A, and Theorem 7 is proved.

Proof of Theorem 8. It is easy to see that for $k \neq l$

$$\mathbf{P}(\nu_k(x) \geqq N, \nu_l(x) \geqq M) \leqq \mathbf{P}(\nu_k(x) \geqq N)\mathbf{P}(\nu_l(x) \geqq M). \tag{5. 11}$$

It follows by Corollary 2 to Lemma C that if $N_1(k)$ is chosen in such a manner that the series

$$\sum_{k=1}^{\infty} \mathbf{P}(\nu_k(x) \geqq N_1(k)) \tag{5. 12}$$

diverges, then $\nu_k(x) \geqq N_1(k)$ for almost all x for an infinity of values of k.

But if

$$N_1(x) = \frac{\log k}{\log\log k} + \frac{\log k\,(\log\log\log k)}{(\log\log k)^2} - g(k)\frac{\log k}{(\log\log k)^2} \tag{5. 13}$$

then

$$\mathbf{P}(\nu_k(x) \geqq N_1(k)) \geqq L_1 \cdot \frac{r_k^{N_1(k)}}{N_1(k)!} \geqq L_2 \frac{e^{\frac{g(k)\log k}{\log\log k}} r_k^{N_1(k)}}{k} \tag{5. 14}$$

where L_1, L_2 are positive constants. Thus the series (5. 12) is divergent provided that

$$g(k) > 2 \log \frac{1}{r_k}. \tag{5. 15}$$

But clearly if $g(k)$ is given such that $g(k) \to +\infty$, the sequence $\{q_n\}$ can be chosen so that r_k should tend to 0 arbitrarily slowly, e. g. that we should have

$$r_k \geqq e^{-\frac{g(k)}{2}} \tag{5. 16}$$

which implies (5. 15). Thus Theorem 8 is proved.

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