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ABOUT AN ESTIMATION PROBLEM OF ZAHORSKI

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Z. Zahorski [4] has asked for the best possible estimation from above of the integral

$$\int_0^{2\pi} |\cos n_1 x + \cos n_2 x + \ldots + \cos n_k x|\, dx,$$

where $0 < n_1 < n_2 < \ldots < n_k$ are integers. He observes that the estimation of $c\sqrt{k}$ is trivial, but he conjectures that $c \log n_k$ is also valid. We shall refute this question twice.

I. *We find a sequence n_i for which*

$$\int_0^{2\pi} \Big| \sum_{i=1}^{k} \cos n_i x \Big|\, dx > c k^{\frac{1}{2}-\varepsilon}.$$

II. *We find a sequence n_i for which*

$$\int_0^{2\pi} \Big| \sum_{i=1}^{k} \cos n_i x \Big|\, dx = \sqrt{\pi}\sqrt{n_k} + o(\sqrt{n_k}),$$

which proves that $O(\sqrt{n_k})$ is the best estimation.

Since the proof of I is much more elementary than the proof of II, we also include it.

The problem remains whether for every sequence $n_1 < n_2 < \ldots < n_k < \ldots$ and for every $\varepsilon > 0$ we have for $k > k_0(\varepsilon)$

$$\int_0^{2\pi} \Big| \sum_{i=1}^{k} \cos n_i x \Big|\, dx < (\sqrt{\pi}+\varepsilon)\sqrt{n_k}.$$

Proof of I. Let us put $n_i = i^2$; $1 \leqslant i \leqslant k$. We are going to prove that

$$\int_0^{2\pi} \Big| \sum_{i=1}^{k} \cos i^2 x \Big|\, dx > c k^{\frac{1}{2}-\varepsilon}. \tag{1}$$

To check this observe that clearly

$$\int_0^{2\pi}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^2dx = \pi k, \tag{2}$$

and it is not difficult to see that for every $\eta > 0$ and $k > k_0(\eta)$

$$\int_0^{2\pi}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^4dx < k^{2+\eta}. \tag{3}$$

Namely, in order to prove (3), observe that

$$\int_0^{2\pi}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^4dx < c_1 \sum_{\substack{i_1^2\pm i_2^2\pm i_3^2\pm i_4^2=0\\ 1\leqslant i_1,i_2,i_3,i_4\leqslant k}} 1 < k^{2+\eta}. \tag{4}$$

Indeed, at least two terms in the sum $i_1^2\pm i_2^2\pm i_3^2\pm i_4^2$ have the same sign. If these terms are i_1^2 and i_2^2, we can write $2\leqslant i_1^2+i_2^2 = \pm i_1^2\pm i_4^2\leqslant 2k^2$. The inequalities $2\leqslant \pm i_3^2\pm i_4^2\leqslant 2k^2$, $1\leqslant i_3, i_4 < k$ have $O(k^2)$ solutions. We denote by $\lambda(x)$ the number of solutions of the equation $i_1^2+i_2^2 = x$. It is well known that $\lambda(x) = o(x^{\varepsilon})$ ([1]). Hence the number of solutions of the equation $i_1^2\pm i_2^2\pm i_3^2\pm i_4^2 = 0$ is

$$k^2 \max_{x=\pm i_3^2\pm i_4^2.} \lambda(x) = o(k^{2+\varepsilon}).$$

From (3) we observe that the set in x for which

$$\Big|\sum_i \cos i^2x\Big| > tk^{1/2}$$

has a measure less than k^{η}/t^4. Thus, a simple computation shows that

$$\int_I\Big(\sum_{i=1}^{k}\cos i^2x)^2dx = \sum_{u=0}^{\infty}\int_{I_u}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^2dx = o(k), \tag{5}$$

where I is the set in which

$$\Big|\sum_{i=1}^{k}\cos i^2x\Big| > k^{\frac{1}{2}+\eta},$$

and the sets I_u are those in which

$$2^u k^{\frac{1}{2}+\eta} < \Big|\sum_{i=1}^{k}\cos i^2x\Big| \leqslant 2^{u+1}k^{\frac{1}{2}+\eta}.$$

([1]) Indeed, $\lambda(x)\leqslant\tau(x)$, where $\tau(x)$ is the number of the divisors of x (see e. g. [2], p. 398), and $\tau(x) = o(x^{\varepsilon})$ (see e. g. [3], p. 26). *(Remark of the Editors).*

Formulae (2) and (5) imply

$$\int_{I'} \left(\sum \cos i^2 x\right)^2 dx = \pi k + o(k), \tag{6}$$

where I' is the complement of I, i. e. for $x \in I'$ we have

$$\left|\sum_{i=1}^{k} \cos i^2 x\right| \leqslant k^{\frac{1}{2}+\eta}.$$

Thus

$$\int_0^{2\pi} \left|\sum_{i=1}^{k} \cos i^2 x\right| dx \geqslant \int_{I'} \left|\sum_{k=1}^{k} \cos i^2 x\right| dx \geqslant \frac{1}{k^{\frac{1}{2}+\eta}} \int_{I'} \left(\sum_{i=1}^{k} \cos i^2 x\right)^2 dx$$

$$= \frac{\pi k + o(k)}{k^{\frac{1}{2}+\eta}} > ck^{\frac{1}{2}-\eta},$$

which completes the proof of I.

The proof of II is based on a theorem of Salem and Zygmund [1]. Let us write

$$S_N = \sum_{1}^{N} \varphi_k(t)(a_k \cos kx + b_k \sin kx),$$

where $\{\varphi_n(t)\}$ is the system of Rademacher functions,

$$c_k^2 = a_k^2 + b_k^2; \quad B_N^2 = \tfrac{1}{2} \sum_{1}^{N} c_k^2,$$

and let $\omega(p)$ be a function of p increasing to $+\infty$ with p, such that $p/\omega(p)$ increases and that $\sum 1/p\,\omega(p) < \infty$. Then, under the assumptions $B_N^2 \to \infty$, $c_N^2 = O\{B_N^2/\omega(B_N^2)\}$, the distribution function of S_N/B_N tends, for almost every t, to the Gaussian distribution with mean value zero and dispersion 1.

Let us set $a_k = 1$, $b_k = 0$ $(k = 1, 2, \ldots)$; then $c_N^2 = 1$, $B_N^2 = \frac{1}{2}N$ where $N = 1, 2, \ldots$ Moreover, it is easy to verify that the function $\omega(p) = \sqrt{p}$ satisfies the conditions of the Salem-Zygmund theorem. Consequently, for almost all t, the distribution function of

$$\frac{S_N}{B_N} = \frac{\sqrt{2}}{\sqrt{N}} \sum_{k=1}^{N} \varphi_k(t) \cos kx$$

tends to the Gaussian distribution with mean value zero and dispersion 1. Furthermore, since the variance of S_N/B_N is equal to 1, we have for almost all t the convergence of the absolute moments of S_N/B_N to the absolute moment of the normalized Gaussian distribution. In other words, we have the relation

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sqrt{2}}{\sqrt{N}}\sum_{k=1}^{N}\varphi_k(t)\cos kx\right|dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|e^{-x^2/2}dx=\sqrt{\frac{2}{\pi}}$$

for almost all t. Hence, using the well-known equality

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\int_0^{2\pi}\left|\sum_{k=1}^{N}\cos kx\right|dx=0,$$

we obtain the relation

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\int_0^{2\pi}\left|\sum_{k=1}^{N}\big(\varphi_k(t)+1\big)\cos kx\right|dx=2\sqrt{\pi} \tag{7}$$

for almost all t.

Let us fix an irrational number t_0 with this property. Let $n_1, n_2, \ldots$ denote the successive indices k for which $\varphi_k(t_0)=1$. Then

$$\sum_{k=1}^{n_N}\big(\varphi_k(t_0)+1\big)\cos kx=2\sum_{k=1}^{N}\cos n_k x$$

and, according to (7),

$$\int_0^{2\pi}\left|\sum_{k=1}^{N}\cos n_k x\right|dx=\sqrt{\pi}\sqrt{n_N}+o(\sqrt{n_N}),$$

which completes the proof of II.

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