Some remarks on Euler's φ function

by

P. ERDÖS (Budapest)

Recently Schinzel [9] proved the following theorem:

Let $a_1, a_2, \ldots, a_k$ be any finite sequence of non-negative integers or infinity. Then there exists an infinite sequence of integers $n_1 < n_2 < \ldots$ such that

$$(1) \qquad \lim_{l\to\infty} \frac{\varphi(n_l+i)}{\varphi(n_l+i-1)} = a_i \quad \text{for} \quad 1 \leqslant i \leqslant k.$$

He also shows that the same result holds for $\sigma(n)$, the sum of divisors of n.

By combining the method of Brun with that of Schinzel I can prove that (1) holds for all multiplicative functions $n^a f(n)$ which satisfy

$$(a) \qquad f(p^a) \to 1 \quad \text{as} \quad p^a \to \infty,$$

$$(b) \qquad \sum_{p_k} |f(p_k^{a_k})-1| = \infty \quad \text{for a certain sequence} \quad a_k \geqslant 1$$

where p runs through the sequence of primes.

I omit the proof, which is not difficult. One can now ask the question whether the conditions (a) and (b) are necessary that (1) should hold. Clearly (b) cannot be dispensed with, since if (b) does not hold then $f(n+1)/f(n)$ is bounded, but it is not clear to what extent (a) is essential, *e. g.*, I cannot decide whether (1) holds for $d(n)$ (the number of divisors of n). In fact I cannot prove the existence of an infinite sequence n_k satisfying

$$d(n_k+1)/d(n_k) \to 1\,(^1).$$

(1) In fact one can conjecture that the quotient $d(n+1)/d(n)$ $(1 \leqslant n < \infty)$ is everywhere dense on the positive real axis. I can prove by Brun's method that $d(n+1)/d(n)$ is dense in a certain interval. The idea of the proof is as follows: Denote by $d'(n)$ the number of divisors of n composed entirely of prime factors $\leqslant n^{1/10}$. It easily follows by Brun's method that $d'(n+1)/d'(n)$ is dense in $(0, \infty)$. Clearly

$$\frac{d(n+1)}{d(n)} \Big/ \frac{d'(n+1)}{d'(n)}$$

can take only a bounded number of possible values. Thus our assertion follows by a simple argument.

Using Brun's method I can prove (1) for $\nu(n)$, where $\nu(n)$ denotes the number of prime factors of n.

Let γ denote Euler's constant, $e^{a} = \prod_{p}(1-1/p)^{-1/p}$ where p runs through all primes. A simple computation shows that $a < \gamma$; $\log_k n$ denotes the logarithm iterated k-times. Now we prove

THEOREM 1. *Let $f(n)$ tend to infinity so that*

$$f(n) \leqslant \log_3 n/\log_6 n + \big(a-\gamma+o(1)\big)\log_3 n/(\log_6 n)^2.$$

Then

$$\varliminf_{n\to\infty}\big(\max_{1\leqslant i\leqslant f(n)} \varphi(n+i)/\min_{1\leqslant j\leqslant f(n)} \varphi(n+j)\big) = 1.$$

Next we show that Theorem 1 is the best possible. In fact we prove

THEOREM 2. *Put*

$$f(n) = \log_3 n/\log_6 n + (c+a-\gamma)\log_3 n/(\log_6 n)^2 \quad (c \geqslant 0).$$

Then

$$\varliminf_{n\to\infty}\big(\max_{1\leqslant i\leqslant f(n)} \varphi(n+i)/\min_{1\leqslant j\leqslant f(n)} \varphi(n+j)\big) = e^{c}.$$

By similar methods I can prove

THEOREM 3[2]. *Let $\lim g(n)/\log_3 n = 0$. Then there exists an infinite sequence n_k such that for all $1 \leqslant i \leqslant g(n_k)$*

$$(2)\quad 1-\varepsilon_k \leqslant \frac{\varphi(n_k+i)}{\varphi(n_k+i-1)} < 1+\varepsilon_k, \quad \textit{where} \quad \varepsilon_k \to 0 \quad \textit{as} \quad k\to\infty.$$

Theorem 3 is the best possible, since it can be shown that if $\lim g(n)/\log_3 n > 0$ then (2) does not hold, and also if $\lim g(n)/\log_3 n = \infty$ then

$$(3)\quad \lim_{n\to\infty}\max_{1\leqslant i\leqslant g(n)} \varphi(n+i)/\varphi(n+i-1) = \infty \quad \text{and}$$

$$\lim_{n\to\infty}\min_{1\leqslant i\leqslant g(n)} \varphi(n+i)/\varphi(n+i-1) = 0.$$

We omit the proof of all these results. It would not be difficult to formulate and prove the analogue of Theorem 2. All these results hold with minor modifications also for $\sigma(n)$.

Denote by $A(n)$ the number of solutions of $\varphi(l) = n$. Several decades ago Carmichael conjectured that there exist no integers with $A(n) = 1$. This conjecture is still unproved and seems very deep. I have corresponded with Kanold and Sierpiński about finding infinitely many integers for which $A(n) = k$. I prove the following

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(2) I stated Theorem 3 incorrectly in my paper [5].

THEOREM 4([3]). *If there exists an integer n with $A(n) = k$ then there exist infinitely many such integers.*

Sierpiński conjectures that for every $k > 1$ there are integers for which $A(n) = k$, and that for every $k > 0$ there are such integers that $\sigma(x) = n$ has k solutions.

Pillai (see P. Erdös [1]) was the first to prove that $\lim A(n) = \infty$, and that for almost all integers $A(n) = 0$. Heilbronn observed (in a letter to Davenport) that

$$\frac{1}{n}\sum_{k=1}^{n} A(k)^2 = \infty.$$

I have proved ([1]) that for a certain $c > 0$ there exists an infinite sequence n_k so that $A(n_k) > n_k^c$ and I have conjectured that the same holds for every $c < 1$( ).

One can conjecture that for $n > n_0(\varepsilon)$

$$\sum_{k=1}^{n} A(k)^2 > n^{2-\varepsilon},$$

but I cannot prove even that $\sum_{k=1}^{n} A(k)^2 > n^{1+\varepsilon}$, though perhaps this is not very difficult. All the results here stated hold also for $\sigma(n)$, and the same unsolved problems remain.

It is not difficult to prove that the inequalities

$$(4)\qquad |\varphi(n+1)-\varphi(n)| < n^c, \quad \text{and} \quad |\sigma(m+1)-\sigma(m)| < m^c$$

both have infinitely many solutions for a certain $c < 1$, but I cannot prove that they have infinitely many solutions for every $c < 1$.

The proof of Theorem 2 is similar to but slightly more complicated than that of Theorem 1; thus for the sake of simplicity we prove only Theorem 1. Denote by $2 = P_1 < P_2 < \ldots < P_k$ the primes not exceeding $f(n)$, by $Q_1 < Q_2 < \ldots < Q_l$ the primes of the interval $(f(n), \frac{1}{2}\log n\rangle$, and by $R_1 < R_2 < \ldots$ the primes greater than $\frac{1}{2}\log n$. Put $A_t = \prod_{i=1}^{t} P_i$, define

$$A_j \leqslant f(n) < A_{j+1}.$$

([3]) Kanold and Sierpiński proved that $A(n) = 2$ for infinitely many integers and Sierpiński found integers n satisfying $A(n) = k$ for many values of k; he did the same for the equation $\sigma(y) = n$. Schinzel [10] proved that $A(n) = 3$ for infinitely many integers n.

([4]) I can prove that the number of solutions of $\varphi(x) = n$ is less than $n\exp(-c\log n\log_3 n/\log_2 n)$ where $\exp z = e^z$ (see [4]).

Now we show that for every ε there exists an $n_0 = n_0(\varepsilon)$, such that for every $n > n_0$ there exist $f(n)$ consecutive integers $m+1, \ldots, m+f(n)$ satisfying $(1 \leqslant i' \leqslant f(n))$

$$\frac{n}{2} < m < n, \tag{5}$$
$$(1-\varepsilon)\prod_{i=1}^{j}\left(1-\frac{1}{P_i}\right) < \frac{\varphi(m+i')}{m+i'} < (1+\varepsilon)\prod_{i=1}^{j}\left(1-\frac{1}{P_i}\right).$$

Clearly (5) will prove Theorem 1.

Define $s_1, s_2, \ldots, s_{f(n)}$ by

$$\prod_{h=1}^{s_1}\left(1-\frac{1}{Q_h}\right) \geqslant \left(1+\frac{\varepsilon}{2}\right)\prod_{i=1}^{j}\left(1-\frac{1}{P_i}\right) > \prod_{h=1}^{s_1+1}\left(1-\frac{1}{Q_h}\right), \tag{6}$$
$$\prod_{h=s_{i-1}+1}^{s_i}\left(1-\frac{1}{Q_h}\right) \geqslant \left(1+\frac{\varepsilon}{2}\right)\prod_{P<P_j}{}^{'}\left(1-\frac{1}{P}\right) > \prod_{h=s_{i-1}+1}^{s_i+1}\left(1-\frac{1}{Q_h}\right),$$

where the $\prod'$ indicates that P runs through the primes $P \leqslant P_j$, $P \nmid i$. It may happen that $s_i = s_{i-1}$. This will in fact be the case if and only if

$$1-\frac{1}{Q_{s_{i-1}+1}} < \left(1+\frac{\varepsilon}{2}\right)\prod_{P<P_j}{}^{'}\left(1-\frac{1}{P}\right). \tag{7}$$

(7) clearly implies

$$\frac{\varphi(i)}{i} < \left(1+\frac{\varepsilon}{2}\right)\prod_{i=1}^{j}\left(1-\frac{1}{P_i}\right)\left(1-\frac{1}{Q_{s_{i-1}+1}}\right)^{-1}. \tag{7'}$$

But since $\sum_{i=1}^{x} i/\varphi(i) < c_1 x$, (7′) and therefore (7) is satisfied only for $o(f(n))$ i's.

First we have to show that the s_i are all defined. Since (7) is satisfied only for $o(f(n))$ i's, we have from (6)

$$\prod_{h=1}^{s_{f(n)}}\left(1-\frac{1}{Q_h}\right) \geqslant (1+o(1))^{f(n)}\left(1+\frac{\varepsilon}{2}\right)^{f(n)}\prod_{i=1}^{j}\left(1-\frac{1}{P_i}\right)^{f(n)}\prod_{i=1}^{f(n)}\frac{i}{\varphi(i)}.$$

Now by a theorem of Mertens[5]

$$\prod_{i=1}^{j}\left(1-\frac{1}{P_i}\right) = (1+o(1))e^{-\gamma}/\log_2 f(n) = (1+o(1))e^{-\gamma}/\log_5 n$$

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[5] The theorem of Mertens in question states that $\prod_{p<y}\left(1-\frac{1}{p}\right) = (1+o(1))\times$ $\times e^{-\gamma}/\log y$ (*e. g.* see [6], p. 351).

and

$$\prod_{i=1}^{f(n)} \frac{i}{\varphi(i)} = \prod_{P \leqslant P_k} \left(1 - \frac{1}{P}\right)^{-[f(n)/P]} = e^{(a+o(1))f(n)}.$$

Thus by a simple calculation

$$\prod_{h=1}^{s_{f(n)}} \left(1 - \frac{1}{Q_h}\right) \geqslant (1+o(1))^{f(n)} \left(1 + \frac{\varepsilon}{2}\right)^{f(n)} \left(\frac{e^{-\gamma}}{\log_5 n}\right)^{f(n)} e^{af(n)}$$

$$\geqslant (1+o(1))^{f(n)} \left(1 + \frac{\varepsilon}{2}\right)^{f(n)} \Big/ \log_2 n.$$

Now, again by the theorem of Mertens([5]) and the definition of the Q's,

$$\prod_{h=1}^{l} \left(1 - \frac{1}{Q_h}\right) = (1+o(1)) \log_4 n / \log_2 n < (1+o(1))^{f(n)} \left(1 + \frac{\varepsilon}{2}\right)^{f(n)} \Big/ \log_2 n$$

or $Q_{s_{f(n)}} < \frac{1}{2} \log n$, whence the s_i, $1 \leqslant i \leqslant f(n)$, are all defined. Put

$$B_i = \prod_{s_{i-1}+1}^{s_i} Q_h, \quad B = \prod_{s_{f(n)}+1}^{l} Q_h \quad (\text{if } s_i = s_{i-1} \text{ then } B_i = 1).$$

Let m satisfy

$$(8) \qquad \begin{aligned} &n/2 < m < n, \quad m \equiv 0 \,(\operatorname{mod}(A_k B)), \\ &m + i \equiv 0 \; (\operatorname{mod} B_i), \quad 1 \leqslant i \leqslant f(n). \end{aligned}$$

Such an m exists, since the moduli are relatively prime and by a well known result on primes (*e. g.* see [6], p. 341; see also [12], p. 56) $\prod_{p<x} p < 4^x$; thus the product of the moduli is less than $4^{(\log n)/2} < n/2$.

Evidently $m+i$ can be divisible by at most $2 \log n / \log_2 n$ R's (since $R > \frac{1}{2} \log n$ and $m + i < 2n$). Thus

$$(9) \qquad \prod_{R|i} \left(1 - \frac{1}{R}\right) > \left(1 - \frac{2}{\log n}\right)^{2\log n/\log_2 n} = 1 + o(1).$$

From (8) it follows that for $P \leqslant P_k$ and $1 \leqslant i \leqslant f(n)$, $P | m+i$ if and only if $P | i$. Thus from (6), (8) and (9)

$$\frac{\varphi(m+i)}{m+i} = (1+o(1)) \frac{\varphi(i)}{i} \left(1 + \frac{\varepsilon}{2}\right) \prod_{P \leqslant P_j}' \left(1 - \frac{1}{P}\right)$$

$$= (1+o(1)) \left(1 + \frac{\varepsilon}{2}\right) \prod_{i=1}^{j} \left(1 - \frac{1}{P_i}\right)$$

if i does not satisfy (7). If i satisfies (7), then from (6), (7), (8) and (9)

$$(1+o(1))\prod_{i=1}^{j}\left(1-\frac{1}{P_i}\right)<\frac{\varphi(m+i)}{m+i}<(1+o(1))\left(1+\frac{\varepsilon}{2}\right)\prod_{i=1}^{j}\left(1-\frac{1}{P_i}\right).$$

Thus in any case (5) is satisfied, which proves Theorem 1.

Now we have to show that Theorem 1 is the best possible. Let

$$f(n)\geqslant\frac{\log_3 n}{\log_6 n}+(c+a-\gamma)\frac{\log_3 n}{(\log_6 n)^2}$$

for some $c>0$; we shall show that

$$\varliminf\left(\max_{1\leqslant i\leqslant f(n)}\varphi(n+i)/\min_{1\leqslant j\leqslant f(n)}\varphi(n+j)\right)>1. \tag{10}$$

At least one of the integers $n+i$, $1\leqslant i\leqslant f(n)$ is divisible by A_j. Thus if (10) were false, there would exist for every $\varepsilon>0$ arbitrarily large integers n such that for all $1\leqslant i\leqslant f(n)$

$$\frac{\varphi(n+i)}{n+i}<(1+\varepsilon)\prod_{P\leqslant P_j}\left(1-\frac{1}{P}\right)<(1+o(1))(1+\varepsilon)e^{-\gamma}/\log_5 n. \tag{11}$$

We have by (9)

$$\frac{\varphi(n+i)}{n+i}=(1+o(1))\prod_{P|n+i}\left(1-\frac{1}{P}\right)\prod_{Q|n+i}\left(1-\frac{1}{Q}\right). \tag{12}$$

Clearly for each Q_i there can be at most one of the numbers $n+1$ $n+2,\ldots,n+f(n)$ which are divisible by Q_i $(Q_i>f(n))$. Thus by (11) (12) and the theorem of Mertens([5])

$$(1+o(1))^{f(n)}(1+\varepsilon)^{f(n)}e^{-\gamma f(n)}/(\log_5 n)^{f(n)}>\prod_{i=1}^{f(n)}\frac{\varphi(n+i)}{n+i}$$

$$\geqslant(1+o(1))^{f(n)}\prod_{P\leqslant Pj}\left(1-\frac{1}{P}\right)^{(f(n)/P+1)}\prod_{h=1}^{l}\left(1-\frac{1}{Q_h}\right)$$

$$=(1+o(1))^{f(n)}e^{-af(n)}/\log_2 n,$$

which, as can be seen by a simple computation, is false for sufficiently small ε. This contradiction proves that Theorem 1 is the best possible.

Proof of Theorem 4. Let $A(n)=k$. We shall prove that for all but $o(x/\log x)$ primes $p<x$ $A[(p-1)n]=k$, and this will clearly prove Theorem 4. If $p>n+1$ and $\varphi(l)=n$ then $\varphi(pl)=(p-1)n$ (since all prime factors of l are $\leqslant n+1$). Thus $A[(p-1)n]\geqslant k$. The solutions $y=pl$ of $\varphi(y)=(p-1)n$ may appropriately be called trivial solutions.

Thus our proof will be complete if we succeed in showing that for every ε and $x > x_0(\varepsilon)$, for all but $\varepsilon x/\log x$ primes $p \leqslant x$, $\varphi(y) = (p-1)n$ has only trivial solutions. First of all we can assume that for (t sufficiently large)

$$(13) \qquad k^2 > t = t_0(\varepsilon), \quad p-1 \not\equiv 0 \pmod{k^2}.$$

To see this we observe that it is well known and follows easily from Brun's method (see [11]) that the number of primes $p \leqslant x$, $p \equiv 1 \pmod{l}$ is less than $c_2 x/\varphi(l)\log(x/l)$. Thus the number of primes $p \leqslant x$ not satisfying (13) is less than

$$\frac{2c_2 x}{\log x} \sum_{k^2 \geqslant t} \frac{1}{k\varphi(k)} + \sum_{x^{1/2} < k^2 \leqslant x} \left(\frac{x}{k^2} + 1\right) < \frac{\varepsilon}{2} \cdot \frac{x}{\log x}$$

(we take first $k^2 \leqslant x^{1/2}$, secondly $k^2 > x^{1/2}$ and use the fact that the number of integers $\equiv 1 \pmod{k^2}$) and less than x is $\leqslant x/k_{\cdot}^2 + 1$.

Now let $x^{1/2} < p \leqslant x$ and let p satisfy (13). Let y be a non trivial solution of $\varphi(y) = (p-1)n$. If y has r distinct prime factors, then clearly $\varphi(y) \equiv 0 \pmod{2^{r-1}}$, and thus $p-1$ is divisible by a square $\geqslant 2^{r-2}/n$. Thus

$$2^{r-2} < nt \quad \text{or} \quad r \leqslant t+n+2.$$

Let $y = q_1^{a_1} \ldots q_r^{a_r}$, $q_1^{a_1} < q_2^{a_2} < \ldots < q_r^{a_r}$, $r \leqslant t+n+2$. Since $y \geqslant p > x^{1/2}$ we have $q_r^{a_r} > x^{(1/2)(t+n+2)}$; also $a_r \leqslant 2$ since otherwise $p-1 = \varphi(y)/n$ would be divisible by a square greater than $(1/n)x^{(1/4)(t+n+2)} > t$ for sufficiently large x, which contradicts (13). Thus there must exist a prime $q > x^\delta$, $\delta = \frac{1}{4}(t+n+2)$ satisfying

$$(14) \quad x^{1/2} < p \leqslant x, \quad q > x^\delta, \quad (p-1)n = 0 \left(\operatorname{mod}(q-1)\right), \quad p \neq q.$$

To complete the proof of Theorem 4 we must show that the number of primes p satisfying (14) is $< (\varepsilon/2)(x/\log x)$. First we prove the following

LEMMA. *The number of solutions of*

$$(15) \qquad (p-1)n = a(q-1), \quad p < x, \quad a < x^{1-\delta}, \quad a \neq n$$

is less than

$$c_3 \frac{x}{a(\log x)^2} \prod_{p \mid a(a-n)} \left(1 + \frac{1}{p}\right)$$

where $c_3 = c_3(n)$ *depends only on* n.

The proof follows easily from Brun's method (*e. g.* see [2], p. 540) and we only outline it. Denote by $r_1, r_2, \ldots$ the primes of the interval

(n, x^{η_1}) where η_1 is sufficiently small. If $q > nx^{\eta_1}$ satisfies (15) we must have

$$q \not\equiv 0 \pmod{r_i}, \quad q \not\equiv 1-n/a \pmod{r_i}, \quad q < nx/a+1 \tag{16}$$

((16) follows from the fact that both p and q are primes and $x^{\eta_1} < p \leqslant x$). If $r \nmid a(a-n)$, then the two residues in (16) are different, and thus we obtain the lemma by a simple application of Brun's method.

Now we split the number of solutions of (14) into three classes. In the first class are the q's greater than $x^{1-\eta_2}$ where η_2 is sufficiently small. Formula (14) then becomes

$$(p-1)n = a(q-1), \quad p < x, \quad 1 \leqslant a \leqslant nx^{\eta_2}.$$

Thus by our lemma the number of solutions of (14) of the first class is (for sufficiently small η_2) less than

$$\frac{c_3 x}{(\log x)^2} \sum_{\substack{1 \leqslant a \leqslant nx^{\eta_2} \\ a \neq n}} \frac{1}{a} \prod_{p \mid a(a-n)} \left(1+\frac{1}{p}\right) < c_4 \eta_2 \frac{x}{\log x} < \frac{\varepsilon}{4} \cdot \frac{x}{\log x}. \tag{17}$$

To prove (17) we observe that

$$\sum_{\substack{a=1 \\ a \neq n}}^{y} \prod_{p \mid a(a-n)} \left(1+\frac{1}{p}\right) \leqslant \sum_{\substack{a=1 \\ a \neq n}}^{y} \left(\prod_{p \mid a} \left(1+\frac{1}{p}\right)^2 + \prod_{p \mid a-n} \left(1+\frac{1}{p}\right)^2 \right)$$

$$< 2 \sum_{a=1}^{y} \prod_{p \mid a} \left(1+\frac{1}{p}\right)^2 + O(1) < 2 \sum_{a=1}^{y} \prod_{p \mid a} \left(1+\frac{3}{p}\right) + O(1)$$

$$< \sum_{d=1}^{\infty} \frac{3^{\nu(d)}}{d} \left[\frac{y}{d}\right] + O(1) < y \sum_{d=1}^{y} \frac{3^{\nu(d)}}{d^2} + O(1) < c_5 y,$$

and (17) follows by partial summation.

The solutions of (14) of the second class are the q's for which $\nu(q-1) < \frac{2}{3} \log_2 x$ $(q < x^{1-\eta_2})$. It follows from Brun's method (see [11]) that the number of primes $p < x$ satisfying

$$p < x, \quad q < x^{1-\eta}, \quad p \equiv 1 \pmod{(q-1)}$$

is less than

$$\frac{c_6 x}{\log x \varphi(q-1)} < \frac{c_7 x \log_2 x}{q \log x} \tag{18}$$

since by a well known result of Landau ([7], p. 218)

$$\varphi(y) > \frac{c_8 y}{\log_2 y}.$$

Thus by (18) the number of solutions of the second class is less than

$$\frac{c_7 x \log_2 x}{\log x} \sum' \frac{1}{q} < \frac{c_9 x \log_2 x}{(\log x)^{1+\eta}} = o\left(\frac{x}{\log x}\right) \tag{19}$$

where the $\sum'$ indicates that $q > x^{\delta}$, $\nu(q-1) < \frac{4}{5}\log_2 q$. Formula (19) follows from the fact (see [1]) that if $q_1 < q_2 < \ldots$ is the sequence of primes satisfying $\nu(q_n - 1) < \frac{4}{5}\log_2 q_n$, then $q_n > n(\log n)^{1+\eta}$.

For the solutions of the third class we have

$$n(p-1) = a(q-1), \quad p < x, \quad x^{\delta} < q < x^{1-\eta_2}, \quad \nu(q-1) > \tfrac{2}{3}\log_2 x. \tag{20}$$

We split the solutions of (20) into two subclasses. In the first subclass are those for which $\nu(a) > \frac{2}{3}\log_2 x$. Here we have

$$\nu(p-1) = \nu(a) + \nu(q-1) - \nu(n) > \tfrac{4}{3}\log_2 x - \nu(n) > \tfrac{5}{4}\log_2 x.$$

It is known (see [1]) that the number of primes $p \leqslant x$ satisfying $\nu(p-1) > (1+\varepsilon)\log_2 x$ is $o(x/\log x)$; consequently the number of solutions of the first subclass is $o(x/\log x)$. The number of solutions of the second subclass is, by our lemma and the theorem of Mertens, less than

$$c_3 \frac{x}{(\log x)^2} \sum' \prod_{p \mid a(a-n)} \left(1 + \frac{1}{p}\right) \Big/ a < c_{10} \frac{x(\log_2 x)^2}{(\log x)^2} \sum' \frac{1}{a} = o\left(\frac{x}{\log x}\right) \tag{21}$$

(the $\sum'$ indicates that $a < x$ and $\nu(a) < \frac{2}{3}\log_2 x$), since

$$\sum \frac{1}{a} < \sum_{k=1}^{[(2/3)\log_2 x]} \left(\sum_{p<x} \frac{1}{p}\right)^k \Big/ k! < \sum_{k=1}^{[(2/3)\log_2 x]} (\log_2 x + c_{11})^k / k! < (\log x)^{9/10}.$$

Thus from (17), (19) and (21) we finally find that the number of solutions of (14) is less than $\frac{1}{2}\varepsilon x/\log x$, and thus Theorem 4 is proved.

By similar but more complicated arguments I can prove that if there exists an integer n with $A(n) = k$, then the number of integers $n \leqslant x$ satisfying $A(n) = k$ is greater than $cx/\log x$ for every c if $x > x_0(c)$.

By more complicated arguments I can prove that for every ε there exists an $A = A(\varepsilon)$ such that the number of primes $p \leqslant x$ satisfying

$$p < x, \quad p \equiv 1 \left(\bmod (q-1)\right), \quad q > A$$

is less than $\varepsilon x/\log x$. Another theorem in this direction is the followi g: Denote by $\nu(k, n)$ the number of prime factors of n not exceedingn$k$; then for every ε there exists an $A = A(\varepsilon)$ such that the number of integers $n \leqslant x$ for which

$$(1-\varepsilon)\log_2 k < \nu(k, n) < (1+\varepsilon)\log_2 k$$

does not hold for some $k > A$ is less than εx. This result is known (see [3]).

Similarly the number of primes $p \leqslant x$ for which

$$(1-\varepsilon)\log_2 k < \nu(k, p-1) < (1+\varepsilon)\log_2 k$$

does not hold for some $k > A$ is less than $\varepsilon x/\log x$.

Finally I can prove that for every ε there exists an $A = A(\varepsilon)$ such that the number of integers $n \leqslant x$ for which $n \equiv 0\,(\mathrm{mod}\,(p-1))$ holds for some $p > A$ is less than εx. From this it is easy to deduce that the density of the integers which can be written as the least common multiple of integers of the form $p^a(p-1)$, $0 \leqslant a$ is 0.

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Reçu par la Rédaction le 24. 9. 1956