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MATHEMATICS

Solution of Two Problems of Jankowska

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In the preceding paper Miss Jankowska puts the following two problems: I. Whether there exist infinitely many pairs of integers a and b satisfying $(a, b) = 1$, $\varphi(a) = \varphi(b)$, $\sigma(a) = \sigma(b)$, $d(a) = d(b)$, where $\varphi(n)$ is Euler's φ function, $\sigma(n)$ is the sum of divisors of n and $d(n)$ is the number of divisors of n. II. Whether for every k there exists a sequence of distinct integers $a_1, a_2, \ldots, a_k$ satisfying

$$\varphi(a_i) = \varphi(a_j), \quad \sigma(a_i) = \sigma(a_j) \quad \text{and} \quad d(a_i) = d(a_j)$$

for all $1 \leqslant i < j \leqslant k$.

Using the methods of one of my earlier papers [1] I am going to solve these problems and also state a few further problems.

First we need three lemmas:

LEMMA 1. *The number of integers not exceeding x all whose prime factors do not exceed $\log x$ is $o(x^{\varepsilon})$ for every $\varepsilon > 0$.*

LEMMA 2. *The number of squarefree integers not exceeding x composed of $c_1 \dfrac{(\log x)^{1+c_2}}{\log\log x}$ arbitrarily given primes not exceeding $(\log x)^{1+c_2}$ is greater than $c_3 x^{\alpha}$, where α is any constant satisfying $0 < \alpha < \dfrac{c_2}{2}$.*

Lemmas 1 and 2 are proved in [1] on pp. 211 and 212.

LEMMA 3. *We can find a constant c_2 so small that for a certain $c_1 > 0$ (in fact we only have to assume $c_1 < 1$) there are more than $c_1(\log x)^{1+c_2}$ primes p not exceeding $(\log x)^{1+c_2}$ such that both $p-1$ and $p+1$ are composed of primes not exceeding $\log x$.*

On p. 212-213 of [1] I proved an analogous lemma, where I required only that all prime factors of $p-1$ be less than $\log x$, but it is clear that the method used there (Brun's method) gives a proof of our Lemma 3.

Now we are ready to solve the problems of Miss Jankowska. Denote by $u_1 < u_2 < \ldots < u_l$ the squarefree integers composed of primes all whose prime factors p do not exceed $(\log x)^{1+c_2}$ and such that all prime factors of $p+1$ and $p-1$ are less than $\log x$. By Lemmas 2 and 3 we obtain that, for sufficiently large x, $l > x^{c_2/4}$. On the other hand, all prime factors of $\varphi(u_i)$ and $\sigma(u_i)$, $1 \leqslant i \leqslant l$ are smaller than $\log x$. Thus, by Lemma 1, there are only $o(x^{\varepsilon})$ different values of $\varphi(u_i)$ and $\sigma(u_i)$ $1 \leqslant i \leqslant l$. The same holds for $d(u_i)$ since it is well known that $d(n) = o(n^{\varepsilon})$ for every $\varepsilon > 0$. Thus, there are $o(x^{3\varepsilon})$ choices for the triplet

$$\{\varphi(u_i), \sigma(u_i), d(u_i)\}, \quad 1 \leqslant i \leqslant l,$$

or there exist r integers $u_{i_1}, u_{i_2}, \ldots, u_{i_r}$ satisfying

$$r \geqslant \frac{l}{x^{3\varepsilon}} > x^{\frac{c_2}{4}-3\varepsilon}; \qquad \varphi(u_{i_1}) = \varphi(u_{i_2}) = \ldots = \varphi(u_{i_r}),\ \sigma(u_{i_1}) = \ldots = \sigma(u_{i_r}); \quad d(u_{i_1}) = \ldots = d(u_{i_r}),$$

which completes the solution of the second problem of Jankowska.

It is clear that by the same method we can prove that for every r there exist k squarefree integers $a_1, a_2, \ldots, a_k$ satisfying

$$d(a_1) = d(a_2) = \ldots = d(a_k) \quad \text{and} \quad a_1 \prod_{p|a_1}\left(1+\frac{j}{p}\right) = a_2 \prod_{p|a_2}\left(1+\frac{j}{p}\right) = \ldots = a_k \prod_{p|a_k}\left(1+\frac{j}{p}\right)$$

for every $-r \leqslant j \leqslant r$, $j \neq 0$. The only change in the proof is that in Lemma 3 we have to require that all prime factors of $p+j$, $-r \leqslant j \leqslant r$, $j \neq 0$ be smaller than $\log x$.

To solve the first problem of Jankowska let a_i, b_i $1 \leqslant i \leqslant k$ satisfy

$$(1) \qquad (a_i, b_i) = 1, \quad \varphi(a_i) = \varphi(b_i), \quad \sigma(a_i) = \sigma(b_i).$$

Our proof will be complete if we succeed in finding another solution a_{k+1}, b_{k+1} of (1). But this is, indeed, easy. Let $v_1 < v_2 < \ldots < v_k \leqslant x$ be the squarefree integers composed of the primes p of Lemma 3, where we further require that $p + \prod_{i=1}^{k} a_i b_i$. Since the last condition disqualifies only a bounded number of primes we obtain, by Lemma 2, that $k > x^{c_2/4}$ and we obtain, just as in the previous proof, two integers v_i and v_j satisfying

$$d(v_i) = d(v_j), \quad \varphi(v_i) = \varphi(v_j), \quad \sigma(v_i) = \sigma(v_j)$$

and no prime factor of $v_i v_j$ divides $\prod_{i=1}^{k} a_i b_i$. Put $(v_i, v_j) = t$. Then $a_{k+1} = \frac{v_i}{t}$, $b_{k+1} = \frac{v_j}{t}$ clearly satisfies (1), and thus the first conjecture of Jankowska is proved.

I conjecture that, for every k, there exists a sequence x_i, $1 \leqslant i \leqslant k$ of distinct integers satisfying

$$(x_i, x_j) = 1,\ 1 \leqslant i < j \leqslant k; \quad \varphi(x_1) = \ldots = \varphi(x_k),\ \sigma(x_1) = \ldots = \sigma(x_k); \quad d(x_1) = \ldots = d(x_k),$$

but I have not yet been able to prove this.

Denote by $A(n)$ the number of solutions of $\varphi(x) = n$. Heilbronn proved (in a letter to Davenport about 25 years ago) that

$$\frac{1}{x} \lim_{x \to \infty} \sum_{n=1}^{k} A^2(n) = \infty .$$

I believe that $\sum_{n=1}^{k} A(n)^2 > x^{2-\varepsilon}$. I have conjectured for a long time that for every $\varepsilon > 0$ and infinitely many n, $A(n) > n^{1-\varepsilon}$, but in [1] I could prove only that, for a certain $c > 0$ and infinitely many n, $A(n) > n^c$.

It is easy to see that if

(2) $\quad (x_i, x_j) = 1, \quad 1 \leqslant i < j \leqslant k \quad$ and $\quad \varphi(x_1) = \varphi(x_2) = \ldots = \varphi(x_k) = n$

then $k \leqslant d(n) < n^{c/\log\log n}$, since all prime factors of the x_i must be of the form $t+1$, $t|n$. On the other hand it can be deduced from results of Prachar [2] and myself that for infinitely many n we can have in (2) $k > n^{c/(\log\log n)^2}$.

Another problem would be to try to estimate the number of solutions in pairs of integers a and b of

(3) $$(a, b) = 1, \quad a < b < n, \quad \varphi(a) = \varphi(b).$$

It seems probable that the number of solutions is $> n^{2-\varepsilon}$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$.

Perhaps I may be permitted to mention the following problem of a different nature:

Can one find for every $\varepsilon > 0$ a sequence of consecutive integers $n+i$, $1 \leqslant i < n^{1-\varepsilon}$ satisfying $\varphi(n+i_1) \neq \varphi(n+i_2)$ for all $0 \leqslant i_1 < i_2 < n^{1-\varepsilon}$. I have not succeeded in solving this problem, not even with $\varepsilon > 1-\delta$ for any $\delta > 0$.

REFERENCES

[1] P. Erdös, Quarterly Journal of Math. **6** (1935), 205-213.

[2] K. Prachar, Monatshefte für Math. **59** (1955), 91-103.

[3] P. Erdös, Quarterly Journal of Math. **7** (1936), 227-229, see also S. Chowla, Proc. Indian Acad. Sci. Section A **5** (1937), 37-39.