ON THE STRUCTURE OF SET-MAPPINGS

By

P. ERDŐS (Budapest), corresponding member of the Academy, and A. HAJNAL (Budapest)

1. Introduction. Let S be a set and $f(x)$ a function which makes correspond to every $x \in S$ a subset $f(x)$ of S so that $x \notin f(x)$. Such a function $f(x)$ we shall call a set-mapping defined on S.

A subset $S' \subseteq S$ is called free (or independent) with respect to the set-mapping $f(x)$, if for every $x \in S'$ and $y \in S'$, $x \notin f(y)$ and $y \notin f(x)$.

Let $\overline{\overline{S}} = m \geqq \aleph_0$ and $n < m$. Assume that $\overline{\overline{f(x)}} < n$ for every $x \in S$. RUZIEWICZ raised the problem if there always exists a free set of power m. Assuming the generalized hypothesis of the continuum the answer to the problem of RUZIEWICZ is positive.[1]

In our present paper we are going to define more general set-mappings and raise analogous questions to those of RUZIEWICZ. Let there be given the set S and a set of its subsets I. Assume that the function $f(X)$ makes correspond to every set X of I a subset $f(X)$ of S so that the intersection of $f(X)$ and X is empty. $f(X)$ will be defined as a set-mapping of S of type I. This is clearly a generalization of the original concept of the set-mapping. (There I consisted of all the subsets of S having one element.) The subset $S' \subseteq S$ is called free (with respect to this set-mapping, or briefly a free set of the set-mapping) if for every $X \subseteq S'$ $(X \in I)$ the intersection of $f(X)$ and S' is empty.

Our aim is the investigation of those set-mappings where I consists either of all subsets of S of a given cardinal t or of all subsets of less than a given cardinal t. In these cases we shall briefly say that the set-mapping is of type t or of type $<t$, respectively. If $\overline{\overline{f(X)}} < n$ for all X of I we shall say that the set-mapping is of order n. Our problems will be of the following kind: Let $\overline{\overline{S}} = m$, further let $f(X)$ be a set-mapping of S of order n and type t. Does there then always exist an independent set of power p?

2. Definitions and notations. In what follows capital letters will denote sets; $x, y, z, \ldots$ are the elements of the sets; greek letters denote ordinals; a, b, c, m, n, t, p denote cardinals and $k, l, s, \ldots$ denote integers.

[1] For the history and older results on this problem see [1], for the more recent results and ramifications see [2].

Union of sets will be denoted by $+$ and Σ, intersection of sets by $\cdot$ and Π. $+$ will also be used to denote addition of ordinals, $-$ will be used to define taking relative complements. $x \in A$ denotes that x is an element of A, $A \subseteq B$ denotes that the set A is contained in B ($A \subset B$ denotes that A is a proper subset of B). ω_α will denote the initial number of $\aleph_\alpha$ ($\omega_0 = \omega$). a^+ will denote the cardinal $\aleph_{\alpha+1}$ where $a = \aleph_\alpha$. If Ω is the initial number of a, then Ω^+ is the initial number of a^+.

The cardinal numbers $b_{a,k}$ are defined by induction as follows:

$$b_{a,0} = a, \quad b_{a,k+1} = 2^{b_{a,k}}.$$

The initial number of $b_{a,k}$ will be denoted by $\Omega_{a,k}$.

The proof of some of our theorems makes use of the generalized continuum hypothesis. These theorems will be denoted by (*).

In the proof of some of our theorems, wich will be denoted by (**), we make use of the following hypothesis:[2]

Let m be a strongly inaccessible cardinal number, $\bar{\bar{S}} = m$. Then one can define a two-valued measure $\mu(X)$ on all subsets of S for which $\mu(S) = 1$; $\mu(\{x\}) = 0$ for all $x \in S$ and the measure is additive for less than m summands. In the present paper we do not investigate the problem, if these theorems are equivalent to the above hypothesis.

Let $\varphi(x)$ be an arbitrary property of the elements of the set H. The set of all $x \in H$ which satisfy $\varphi(x)$ will be denoted by $\{x : \varphi(x)\}$. The notation $\{\ \}$ will also be used to denote sets whose elements are those which are contained in the brackets $\{\ \}$. The set $\{X : X \subseteq S; \bar{\bar{X}} = t\}$ will be denoted by $[S]^t$ and the set $\{X : X \subseteq S; \bar{\bar{S}} < t\}$ by $[S]^{<t}$.

For the study of the set-mappings we introduce the notations $(m, n, t) \rightarrow p$ and $(m, n, <t) \rightarrow p$ to denote that every set-mapping, defined on S ($\bar{\bar{S}} = m$), of order n and type t (or $<t$, respectively) has a free set of power p. In the opposite case we write $(m, n, t) \nrightarrow p$ and $(m, n, <t) \nrightarrow p$, respectively.

3. A short summary of the principal results and problems. As a first step we show that if the type t of the set-mapping is infinite, one can never assure the existence of a (non-trivial) free set, even if the order of the set-mapping is 2, in other words we shall show that for every m and $t \geqq \aleph_0$, $(m, 2, t) \nrightarrow t$. (From this it is easy to deduce that $(m, 2, <t) \nrightarrow \aleph_0$ for $t > \aleph_0$ (see Theorem 1).)

Thus we can expect positive results only for $(m, n, k) \rightarrow p$ and $(m, n, <\aleph_0) \rightarrow p$, for simplicity we write $(m, n, \omega) \rightarrow p$ instead of $(m, n, <\aleph_0) \rightarrow p$.

For the symbol $(m, n, \omega) \rightarrow p$ we obtain the following negative result: Let $m < \aleph_\omega$, then $(m, 2, \omega) \nrightarrow \aleph_0$ (see Theorem 2).

[2] See [3] and [4].

Then we prove the following surprising result: (**) Let m be strongly inaccessible and $n < m$, then $(m, n, \omega) \rightarrow m$ (see Theorem 7).

The simplest unsolved problem here is the following:

PROBLEM 1. $(\aleph_\omega, 2, \omega) \rightarrow \aleph_0$?

(It is easy to see that $(\aleph_\omega, 2, \omega) \not\rightarrow \aleph_1$, this easily follows from $(m, 2, \omega) \not\rightarrow \aleph_0$ for $m < \aleph_\omega$).

The problem of the set-mappings of type ω is closely connected with a problem considered by ERDŐS—RADO:

Can one split for each k $(1 \leqq k < \aleph_0)$ the subsets of k elements of S into two classes so that if $S_1 \subseteq S$ $(\overline{\overline{S_1}} \geqq \aleph_0)$ is an arbitrary infinite subset of S, then there always exists a k such that S_1 has a subset of k elements in both classes?[3]

By the methods used in this paper we prove that if m is less than the first strongly inaccessible cardinal m_0, $m_0 > \aleph_0$, such a splitting is possible, but if m is strongly inaccessible, $m > \aleph_0$, then there always exists an $S_1 \subseteq S$, $\overline{\overline{S_1}} = m$, such that for every k $(1 \leqq k < \aleph_0)$ all subsets of S_1 having k elements are in the same class (here we have to use (**)). (See Theorem 9.) By the symbolism introduced in [6] these results can be expressed in the form:

$m \not\rightarrow (\aleph_0)^{<\aleph_0}$ if m is less than the first strongly inaccessible cardinal $m_0 > \aleph_0$ and

$m \rightarrow (m)^{<\aleph_0}$ if m is a strongly inaccessible cardinal, $m > \aleph_0$.

For the set-mappings of finite type our results are more complete.

For the set-mappings of type 1 we already stated that (*) $(m, n, 1) \rightarrow m$ for $n < m$ and $m \geqq \aleph_0$.

For strongly inaccessible cardinals m we have $(m, n, \omega) \rightarrow m$ (**) and consequently $(m, n, k) \rightarrow m$ $(n < m)$ for every k too.

If $m = \aleph_\alpha$, where α is a limit number (but we assume that $\aleph_\alpha$ is not inaccessible), we can prove, using (*), that $(m, n, k) \rightarrow m$ $(n < m)$ (see Theorem 8).

Thus in these cases the exact analogue of the results of RUZIEWICZ for $k = 1$ holds.

If $m = \aleph_{\alpha+1}$, the analogue of the conjecture of RUZIEWICZ fails already for $k \geqq 2$, i. e. we shall show that $(\aleph_{\alpha+k-1}, \aleph_\alpha, k) \not\rightarrow k+1$ for $k = 1, 2, \ldots$ (see Lemma 2).

On the other hand, we can prove that (*) $(\aleph_{\alpha+k}, \aleph_\alpha, k) \rightarrow \aleph_{\alpha+1}$ (see Theorem 3).

[3] See [5].

Thus we know that the smallest m, for which the symbols $(m, \aleph_\alpha, k) \to \aleph_0, \ldots, (m, \aleph_\alpha, k) \to \aleph_{\alpha+1}$ are true, is $\aleph_{\alpha+k}$, but, on the other hand, we do not know the greatest p for which $(\aleph_{\alpha+k}, \aleph_\alpha, k) \to p$ is true. We can prove only the following negative result: (*) $(\aleph_{\alpha+k}, \aleph_\alpha, k) \not\to \aleph_{\alpha+k}$ if $k \geqq 2$ (see Theorem 5).

So the simplest unsolved problem here is the following:

PROBLEM 2. $(\aleph_3, \aleph_0, 3) \to \aleph_2$?

($(\aleph_3, \aleph_0, 3) \to \aleph_1$ is true and $(\aleph_3, \aleph_0, 3) \to \aleph_3$ is false.)

For the set-mappings of finite order we have the following results: If m is infinite, then $(m, l+1, k) \to \aleph_0$ (see Theorem 10) and (*) $(\aleph_{\alpha+k-1}, l+1, k) \to \aleph_\alpha$ for $k = 1, 2, \ldots;\ l = 1, 2, \ldots$ (see Theorem 11).

We have no negative results corresponding to the result of Lemma 2, but we know that (*) $(\aleph_{\alpha+1}, 2, 2) \not\to \aleph_{\alpha+1}$ (see Theorem 5).

The simplest unsolved problem here is the following:

PROBLEM 3. $(\aleph_2, 2, 3) \to \aleph_1$?

($(\aleph_2, 2, 3) \to \aleph_0$ is true, $(\aleph_2, 2, 3) \to \aleph_2$ is false.)

We investigate separately the set-mappings defined on a finite set S. We have the following result: If p, m, l and k are integers and $p(m, l, k)$ denotes the greatest integer p for which $(m, l+1, k) \to p$ is true, then

$$c_1 \sqrt[k+1]{m} < p(m, l, k) < c_2 \sqrt[k]{m \log m}$$

where the positive real numbers c_1 and c_2 depend on k and l.

The following problem arises:

PROBLEM 4. *What is the exact order of magnitude of* $p(m, l, k)$?

4. Proof of the results. We enumerate some simple properties of our symbols:

If $m < m'$ and $(m, n, t) \to p$, then $(m', n, t) \to p$.
If $n < n'$ and $(m, n', t) \to p$, then $(m, n, t) \to p$.
If $p < p'$ and $(m, n, t) \to p'$, then $(m, n, t) \to p$.

Similar theorems are true for the symbol $(m, n, < t) \to p$ and we also have

$$(m, n, t) \to p \quad \text{and} \quad (m, n, < t) \to p \quad \text{if} \quad t < t' \quad \text{and} \quad (m, n, < t') \to p.$$

In what follows we shall use these theorems without references.

LEMMA 1. *Let S be a set, $\overline{\overline{S}} \geqq t \geqq \aleph_0$. There exists a function $g(X)$ defined on the set $[S]^t$ which satisfies the following conditions:*

$$(1)\ g(X) \subset X, \quad (2)\ \overline{\overline{g(X)}} = t, \quad (3)\ g(X) \neq g(Y) \quad \textit{if} \quad X \neq Y.$$[4]

[4] This construction is due to J. NOVAK.

PROOF. First we prove the existence of such a function in the case $\overline{\overline{S}} = t$.

Then $\overline{\overline{[S]^t}} = 2^t$. Let Ω denote briefly the initial number of 2^t, and let $\{X_\nu\}_{\nu<\Omega}$ be a well-ordering of $[S]^t$ of type Ω. We define the function $g(X_\nu)$ by transfinite induction as follows: Let $g(X_0)$ be an arbitrary proper subset of power t of X_0. Suppose that $g(X_\mu)$ is already defined for all $\mu < \nu$ where $\nu < \Omega$. Then $\overline{\overline{\{g(X_\mu)\}_{\mu<\nu}}} \leqq \overline{\overline{\nu}} < 2^t$ and, on the other hand, $\overline{\overline{[X_\nu]^t}} = 2^t$. Thus there exist proper subsets of power t of X_ν different from all $g(X_\mu)$ for all $\mu < \nu$. Let $g(X_\nu)$ be such a subset of X_ν with the smallest subscript. Thus $g(X_\nu)$ is defined for all $\nu < \Omega$ and it is obvious that conditions (1), (2) and (3) hold.

Now let us consider the case $\overline{\overline{S}} > t$. Let M be a maximal subset of $[S]^t$ such that $\overline{\overline{X}} = t$ for every $X \in M$ and $\overline{\overline{X \cdot Y}} < t$ if X, Y are two distinct elements of M. (The existence of such an M is assured by ZORN's lemma.) Let $\{Y_\alpha\}_{\alpha<\tau}$ be a well-ordering of M. Since $\overline{\overline{Y_\alpha}} = t$ for every $\alpha < \tau$, we already know that there exist functions $g_\alpha(X)$ defined on the set $[Y_\alpha]^t$ satisfying the conditions (1), (2) and (3). By the definition of M there exists an α ($\alpha < \tau$), for every $X \in [S]^t$, for which $\overline{\overline{X \cdot Y_\alpha}} = t$. Let $\alpha(X)$ denote the smallest α for which $\overline{\overline{X \cdot Y_\alpha}} = t$.

We define $g(X)$ as follows:

$$g(X) = g_{\alpha(X)}(X \cdot Y_{\alpha(X)}) + (X - Y_{\alpha(X)}).$$

From the properties of the functions $g_\alpha(X)$ it follows immediately that $g(X)$ satisfies the conditions (1) and (2).

We have only to prove $g(X_1) \neq g(X_2)$ for $X_1 \neq X_2$ where $X_1, X_2 \in [S]^t$. We distinguish two cases: (i) $\alpha(X_1) = \alpha(X_2) = \alpha$, (ii) $\alpha(X_1) \neq \alpha(X_2)$.

(i) Then either $X_1 \cdot Y_\alpha \neq X_2 \cdot Y_\alpha$ or $X_1 - Y_\alpha \neq X_2 - Y_\alpha$ and therefore either $g_\alpha(X_1 \cdot Y_\alpha) \neq g_\alpha(X_2 \cdot Y_\alpha)$ or $X_1 - Y_\alpha \neq X_2 - Y_\alpha$, hence $g(X_1) \neq g(X_2)$.

(ii) We may suppose $\alpha(X_1) < \alpha(X_2)$. Then, by the definition of $\alpha(X)$,

$$\overline{\overline{g(X_1) \cdot Y_{\alpha(X_1)}}} = t \quad \text{and} \quad \overline{\overline{g(X_2) \cdot Y_{\alpha(X_1)}}} < t,$$

hence $g(X_1) \neq g(X_2)$. Q. e. d.

REMARK. In case $\overline{\overline{S}} > t$, we can prove with a slight modification of the construction the existence of a function $g_1(X)$ defined on the set $[S]^t$ satisfying the conditions (1), (2), (3) and the following condition too:

(4) For every $Y \in [S]^t$ there is an $X \in [S]^t$ for which $Y = g_1(X)$.[5]

We can not solve the following problem:

PROBLEM 5. *Let S be a set, $\overline{\overline{S}} > t \geqq \aleph_0$. Does there exist a function $g_2(X)$ defined on the set $[S]^t$ satisfying the conditions* (2), (3), (4) *and the following stronger condition* (1′) *instead of* (1):

[5] See ERDŐS—FODOR—HAJNAL's forthcoming paper.

(1′) *For all* $X \in [S]^t$ $g(X) \subseteq X$ *and* $\overline{\overline{X-g(X)}} = t$?

THEOREM 1. $(m, 2, t) \not\to t$ *if* $t \geqq \aleph_0$; $(m, 2, <t) \not\to \aleph_0$ *if* $t > \aleph_0$.

We shall only prove the first statement; the second is a consequence of it.

PROOF. Let S be a set, $\overline{\overline{S}} = m \geqq t$. We define a set-mapping $f(X)$ of S of type t and order 2, having no free sets of power t. Let $g(X)$ be a function defined on the set $[S]^t$ satisfying the conditions (1), (2), (3) of Lemma 1. We define the set-mapping $f(X)$ as follows: Let X be an arbitrary element of $[S]^t$. If there is a $Y \in [S]^t$ for which $X = g(Y)$, then by condition (3) this Y is uniquely determined and by (1) the set $Y - X$ is not empty. In this case we choose an element x of $Y - X$ and we put $f(X) = \{x\}$. In the other case we put $f(X) = 0$. It is obvious that $f(X)$ is a set-mapping of S of type t and order 2.

If X_0 is an arbitrary element of $[S]^t$, then by (1) and (2) we have $g(X_0) \subset X_0$, $g(X_0) \in [S]^t$.

By the definition of $f(X), f(g(X_0)) \cdot X_0 = \{x_0\} \neq 0$. It follows that X_0 is not free. Q. e. d.

We need the following lemma:

LEMMA 2. $(\aleph_{\alpha+k-1}, \aleph_\alpha, k) \not\to k+1$; $(\aleph_{\alpha+k}, \aleph_\alpha, k) \to k+1$ $(k = 1, 2, \ldots)$.

Lemma 2 is another form of a theorem of KURATOWSKI—SIERPINSKI proved in [7]. Therefore we omit the proof of it.[6]

The proof of KURATOWSKI shows that the first part of Lemma 2 is true in the following stronger form too:

Let S be a set of power $\aleph_{\alpha+k-1}$ and $\{x_\nu\}_{\nu<\omega_{\alpha+k-1}}$ a well-ordering of S of type $\omega_{\alpha+k-1}$. One can define a set-mapping $f(X)$ of S of type k and order $\aleph_\alpha$, having no free sets of power $k+1$ and satisfying the following condition: (1) *For every* $\{x_{\nu_1}, \ldots, x_{\nu_k}\} \in [S]^k$, $x_\gamma \in f(\{x_{\nu_1}, \ldots, x_{\nu_k}\})$ implies that $\gamma < \operatorname{Max}(\nu_1, \ldots, \nu_k)$.

THEOREM 2. $(m, 2, \omega) \not\to \aleph_0$ *if* $m < \aleph_\omega$.

PROOF.[7] Let S be a set of power $\aleph_k$ $(k = 1, 2, \ldots)$; $\{x_\nu\}_{\nu<\omega_k}$ a well-ordering of S and $f(X)$ a set-mapping of S of type k and order $\aleph_1$ satisfying the first part of Lemma 2.

[6] The equivalence of Lemma 2 and of KURATOVSKI's paper [7] may be seen by using the following idea. The splitting of Z^{n+1} in [7] induces a set-mapping $f(x_1, \ldots, x_n)$ as follows: $f(x_1, \ldots, x_n) = \sum_{(i_1, \ldots, i_n)} \sum_{k=1}^{n+1} \{x : (x_{i_1}, \ldots, x_{i_{k-1}}, x, x_{i_{k+1}}, \ldots, x_{i_n}) \in A_k\}$, where $(i_1, \ldots, i_n)$ runs over all permutations of the numbers $1, \ldots, n$.

[7] The idea of this proof for the case $k = 0$ is due to J. SURÁNYI.

We define a set-mapping $g(X)$ of S of type ω and order 2 as follows:

Let X be an arbitrary element of $[S]^{<\aleph_0}$ and $\{f_l(X)\}_{l<\omega}$ a well-ordering of type ω of the set $f(X)$. We have two cases: a) X has at most k elements. Then we put $g(X)=0$. b) X has more than k elements. Then X has the form $\{x_{\nu_1},\ldots,x_{\nu_k},x_{\nu_{k+1}},\ldots,x_{\nu_{k+l}}\}$ ($\nu_i<\nu_j$ for $i<j$).

Let $g(X)=\{f_l(\{x_{\nu_1},\ldots,x_{\nu_k}\})\}$.

Since $g(X)\cdot X=0$, by condition (1) of Lemma 2, $g(X)$ is a set-mapping. It is obvious that $g(X)$ is of type ω and order 2. We have only to prove that $g(X)$ has no infinite free sets.

Let Y be an arbitrary infinite subset of S. Then there is a subset $\{x_{\nu_s}\}_{s<\omega}$ of Y such that $\nu_{s_1}<\nu_{s_2}$ for $s_1<s_2$. The set-mapping $f(X)$ has no free sets of $k+1$ elements. Therefore there is an i $(0\leqq i\leqq k)$ and an $l<\omega$ such that

$$x_{\nu_i}=f_l(\{x_{\nu_0},\ldots,x_{\nu_{i-1}},x_{\nu_{i+1}},\ldots,x_{\nu_k}\}).$$

Let X_0 denote the set $\{x_{\nu_0},\ldots,x_{\nu_{i-1}},x_{\nu_{i+1}},\ldots,x_{\nu_{k+l}}\}$. Then $X_0\in[S]^{<\aleph_0}$, $X_0\subset Y$ and by b) $g(X_0)=\{x_{\nu_i}\}$. Thus $g(X_0)\cdot Y=\{x_{\nu_i}\}\neq 0$; hence Y is not free. Q. e. d.

LEMMA 3. *If* $[S]^{k+1}=\sum_{\nu<\omega_\alpha} I_\nu$ *and* $\bar{\bar{S}}>b_{\aleph_\alpha,k}$, *then there exists a subset* S_0 *of* S *and a* $\nu_0<\omega_\alpha$ *such that* $[S_0]^{k+1}\subseteq I_{\nu_0}$ *and* $\bar{\bar{S}}_0\geqq\aleph_{\alpha+1}$ $(k=0,1,2,\ldots)$.[8]

PROOF. Without loss of generality we may assume that the sets I_ν are mutually exclusive.

We prove the theorem by induction on k. In the case $k=0$ the theorem is obviously true. Suppose now that it is true for a certain k and let S be a set satisfying the following conditions:

$$(')\ \bar{\bar{S}}>b_{\aleph_\alpha,k+1},\quad ('')\ [S]^{k+2}=\sum_{\nu<\omega_\alpha} I_\nu,\quad (''')\ I_{\nu_1}\cdot I_{\nu_2}=0\quad\text{for}\quad \nu_1\neq\nu_2.$$

Let $x_0,\ldots,x_k$ be $k+1$ arbitrary elements of S. We split the set $S-\{x_0,\ldots,x_k\}$ into classes. The elements x and y belong to the same class if and only if there is a $\nu<\omega_\alpha$ such that $\{x,x_0,\ldots,x_k\}\in I_\nu$ and $\{y,x_0,\ldots,x_k\}\in I_\nu$. It follows from the conditions ($''$) and ($'''$) that this really defines a splitting of the set $S-\{x_0,\ldots,x_k\}$ into classes. Let Q_{β_1} denote these different classes where β_1 runs through the ordinals less than ω_α. We select an element x_{β_1} from each of the non-empty classes.

Suppose that we have already defined the classes $Q_{\beta_1\ldots\beta_\mu}$ and the elements $x_{\beta_1\ldots\beta_\mu}$ for $\mu<\lambda$. Let $\{\beta_\mu\}_{\mu<\lambda}$ be a sequence of type λ of these ordi-

[8] This theorem is proved in [6]. By the symbolism introduced there the theorem can be expressed in the form: If $m>b_{\aleph_\alpha,k}$, then $m\to(\aleph_{\alpha+1})^{k+1}_{\aleph_\alpha}$.

nals. Let us consider the classes

$$\prod_{\mu<\lambda} Q_{\beta_1\ldots\beta_\mu} - \{x_0,\ldots,x_k, x_{\beta_1}\ldots x_{\beta_\mu}\}_{\mu<\lambda} = Q'_{\{\beta_\mu\}_{\mu<\lambda}}.$$

We split these classes into subclasses as follows: The elements x, y of $Q'_{\{\beta_\mu\}_{\mu<\lambda}}$ are in the same class if and only if for an arbitrary $A \in [\{x_0,\ldots,x_k, x_{\beta_1\ldots\beta_\mu}\}_{\mu<\lambda}]^{k+1}$ there exists a $\nu < \omega_\alpha$ such that the sets $\{x\}+A, \{y\}+A$ belong to the same class I_ν. It follows from the conditions ($''$) and ($'''$) that this really defines a splitting of the class $Q'_{\{\beta_\mu\}_{\mu<\lambda}}$ into subclasses. Let $Q_{\beta_1\ldots\beta_\mu\ldots\beta_\lambda}$ denote the classes thus obtained where β_λ runs over the ordinals less than a certain initial number. We select an element $x_{\beta_1\ldots\beta_\lambda}$ from the non-empty classes $Q_{\beta_1\ldots\beta_\lambda}$.

We shall prove that there is a sequence $\{\beta_\lambda\}_{\lambda<\Omega^+_{\aleph_\alpha,k}}$ for which the elements $x_{\beta_1\ldots\beta_\lambda}$ are really defined. First of all we prove that $\beta_\lambda < \Omega_{\aleph_\alpha,k+1}$ for every $\lambda < \Omega^+_{\aleph_\alpha,k}$ and for every sequence $\{\beta_\mu\}_{\mu<\lambda}$, i. e. at every step in our process the power of the set of all non-empty subclasses of the class $Q'_{\{\beta_1\ldots\beta_\mu\ldots\}_{\mu<\lambda}}$ is at most $b_{\aleph_\alpha,k+1}$. Namely, we can obtain these classes as follows: First we split the set $Q'_{\{\beta_1\ldots\beta_\mu\ldots\}_{\mu<\lambda}}$ corresponding to an element A of $[\{x_0,\ldots,x_k, x_{\beta_1\ldots\beta_\mu}\}_{\mu<\lambda}]^{k+1}$ into $\aleph_\alpha$ classes and then we say that x and y belong to the same class if they belong to the same class for all A. On the other hand, $\bar{\lambda} \leqq b_{\aleph_\alpha,k}$ and so $\overline{[\{x_0,\ldots,x_k, x_{\beta_1\ldots\beta_\mu}\}_{\mu<\lambda}]^{k+1}} \leqq b_{\aleph_\alpha,k}$. Thus the power of the set of all non-empty subclasses of $Q'_{\{\beta_\mu\}_{\mu<\lambda}}$ is at most $\aleph_\alpha^{b_{\aleph_\alpha,k}} = 2^{b_{\aleph_\alpha,k}} = b_{\aleph_\alpha,k+1}$.

Let A_λ denote the set of all sequences $\{\beta_\mu\}_{\mu<\lambda}$. By the result just proved we have

$$\bar{A}_\lambda \leqq b^{\bar{\lambda}}_{\aleph_\alpha,k+1} \leqq b^{b_{\aleph_\alpha,k}}_{\aleph_\alpha,k+1} = 2^{b_{\aleph_\alpha,k}\cdot b_{\aleph_\alpha,k}} = b_{\aleph_\alpha,k+1} \text{ if } \lambda < \Omega^+_{\aleph_\alpha,k}.$$

Let S_λ denote the set $\sum_{\{\beta_\mu\}_{\mu<\lambda}\in A_\lambda} \{x_0,\ldots,x_k, x_{\beta_1\ldots\beta_\mu}\}_{\mu<\lambda}$. If $\lambda < \Omega^+_{\aleph_\alpha,k}$, then $\bar{S}_\lambda = \bar{A}_\lambda\cdot\bar{\lambda} \leqq b_{\aleph_\alpha,k+1}$.

It is obvious from the construction that $S = S_{\Omega^+_{\aleph_\alpha,k}} + \sum_{\{\beta_\lambda\}_{\lambda<\Omega^+_{\aleph_\alpha,k}}\in A_{\Omega^+_{\aleph_\alpha,k}}} \left(\prod_{\lambda<\Omega^+_{\aleph_\alpha,k}} Q_{\beta_1\ldots\beta_\lambda}\right)$ and $S_{\Omega^+_{\aleph_\alpha,k}} = \sum_{\lambda<\Omega^+_{\aleph_\alpha,k}} S_\lambda$.

Therefore $\bar{\bar{S}}_{\Omega^+_{\aleph_\alpha,k}} \leqq \overline{\Omega^+_{\aleph_\alpha,k}} \cdot b_{\aleph_\alpha,k+1} \leqq b_{\aleph_\alpha,k+1}$. It follows by condition ($'$) that there is a sequence $\{\beta_\lambda\}_{\lambda<\Omega^+_{\aleph_\alpha,k}}$ for which $\prod_{\lambda<\Omega^+_{\aleph_\alpha,k}} Q_{\beta_1\ldots\beta_\lambda}$ is non-empty. Thus $Q_{\beta_1\ldots\beta_\lambda}$ is non-empty and therefore $x_{\beta_1\ldots\beta_\lambda}$ is really defined. It is obvious from the construction that if $\lambda_1 \neq \lambda_2$, then $x_{\beta_1\ldots\beta_{\lambda_1}} \neq x_{\beta_1\ldots\beta_{\lambda_2}}$.

For the sake of brevity let $x_{k+\lambda}$ denote $x_{\beta_1 \ldots \beta_\lambda}$. Then for the set $S_1 = \{x_\lambda\}_{\lambda < \Omega^+_{\aleph_\alpha, k}}$ we get $\overline{\overline{S}}_1 > b_{\aleph_\alpha, k}$. If $\{x_{\lambda_0}, \ldots, x_{\lambda_k}\}$ is an arbitrary subset of $k+1$ elements of the set S_1, then by the construction there is exactly one $\nu < \omega_\alpha$ such that $\{x_{\lambda_0}, \ldots, x_{\lambda_k}, x_\lambda\} \in I_\nu$ for every $\lambda > \mathrm{Max}(\lambda_0, \ldots, \lambda_k)$. In this case we say that $\{x_{\lambda_0}, \ldots, x_{\lambda_k}\}$ is an element of I'_ν. It is obvious that $[S_1]^{k+1} = \sum_{\nu < \omega_\alpha} I'_\nu$ and $I'_{\nu_1} \cdot I'_{\nu_2} = 0$ for $\nu_1 \neq \nu_2$. Thus by the induction hypothesis there is a set $S_0 \subseteq S_1$ and a $\nu_0 < \omega_\alpha$ such that $[S_0]^{k+1} \subseteq I'_{\nu_0}$ and $\overline{\overline{S}}_0 \geqq \aleph_{\alpha+1}$.

But if $\{x_{\lambda_0}, \ldots, x_{\lambda_{k+1}}\}$ $(\lambda_0 < \cdots < \lambda_{k+1})$ is an arbitrary subset of $k+2$ elements of the set S_0, then $\{x_{\lambda_0}, \ldots, x_{\lambda_k}\} \in I'_{\nu_0}$ and therefore $\{x_{\lambda_0}, \ldots, x_{\lambda_{k+1}}\} \in I_\nu$, i. e. $[S_0]^{k+2} \subseteq I_{\nu_0}$. Thus we have finished the induction, and Lemma 3 is proved. Q. e. d.

LEMMA 4. *Let S be a set, $\overline{\overline{S}} = m \geqq \aleph_0$ $(n < m)$ and $f(X)$ a set-mapping of S of type* 1 *and order n. The set S is the sum of at most n free sets.*

Lemma 4 is a theorem of G. FODOR.[9]

(*) THEOREM 3. $(\aleph_{\alpha+k}, \aleph_\alpha, k) \to \aleph_{\alpha+1}$ $(k = 1, 2, \ldots)$.

PROOF. Let S be a set, $\overline{\overline{S}} = \aleph_{\alpha+k}$ and $f(X)$ a set-mapping of S of type k and order $\aleph_\alpha$. We have to prove the existence of a free set of power $\aleph_{\alpha+1}$. For $k = 1$ the theorem is well known. We shall suppose $k > 1$.

$f(\{x_1, \ldots, x_k\})$ will be denoted briefly by $f(x_1, \ldots, x_k)$. Let $\{x_1, \ldots, x_{k-1}\}$ be an arbitrary element of $[S]^{k-1}$. We define the set-mapping $g_{x_1, \ldots, x_{k-1}}(X)$ of the set $S - \{x_1, \ldots, x_{k-1}\}$ of type 1 and order $\aleph_\alpha$ as follows:

For every $x \in S - \{x_1, \ldots, x_{k-1}\}$ let $g_{x_1 \ldots x_{k-1}}(\{x\}) = f(x_1, \ldots, x_{k-1}, x)$. By Lemma 4 $S - \{x_1, \ldots, x_{k-1}\} = \sum_{\nu < \omega_\alpha} S^\nu_{x_1 \ldots x_{k-1}}$ for every $x_1, \ldots, x_{k-1}$, where the sets $S^\nu_{x_1 \ldots x_{k-1}}$ are free sets of the set-mapping $g_{x_1 \ldots x_{k-1}}(X)$ for every $\nu < \omega_\alpha$. We may suppose that $S^{\nu_1}_{x_1 \ldots x_{k-1}} \cdot S^{\nu_2}_{x_1 \ldots x_{k-1}} = 0$ for $\nu_1 \neq \nu_2$. Let $\{x_\mu\}_{\mu < \omega_{\alpha+k}}$ be a well-ordering of S of type $\omega_{\alpha+k}$. Let $\{x_{\mu_1}, \ldots, x_{\mu_k}\}$ $(\mu_1 < \cdots < \mu_k)$ be an arbitrary subset of k elements of the set S. We split the set $[S]^k$ into subsets $I_{(\nu_1 \ldots \nu_k)}$ where the symbol $(\nu_1 \ldots \nu_k)$ used as subscript consists of k ordinals less than ω_α. $\{x_{\mu_1}, \ldots, x_{\mu_k}\} \in I_{(\nu_1 \ldots \nu_k)}$ if and only if $x_{\mu_i} \in S^{\nu_i}_{x_{\mu_1} \ldots x_{\mu_{i-1}} x_{\mu_{i+1}} \ldots x_{\mu_k}}$ for every i $(1 \leqq i \leqq k)$. Obviously

$$[S]^k = \sum_{(\nu_1 \ldots \nu_k)\ (\nu_i < \omega_\alpha;\ i = 1, \ldots, k)} I_{(\nu_1 \ldots \nu_k)}.$$

The set of the symbols $(\nu_1 \ldots \nu_k)$ is of power $\aleph_\alpha$ and by (*) $\overline{\overline{S}} > b_{\aleph_\alpha, k-1}$, thus by Lemma 3 there is a subset S_0 and a symbol $(\nu^0_1 \ldots \nu^0_k)$ such that $[S_0]^k \subseteq I_{(\nu^0_1 \ldots \nu^0_k)}$ and $\overline{\overline{S}}_0 \geqq \aleph_{\alpha+1}$.

[9] See [8], Theorem 1.

The set S_0 is free. It is sufficient to prove that if $\{x_{\mu_0}, \dots, x_{\mu_k}\}$ $(\mu_0 < \dots < \mu_k)$ is an arbitrary subset of $k+1$ elements of the set S_0, then $x_{\mu_i} \notin f(x_{\mu_0}, \dots, x_{\mu_{i-1}}, x_{\mu_{i+1}}, \dots, x_{\mu_k})$ for $i = 0, \dots, k$.

In fact, for example, in the cases $i \neq 0$ we have $\{x_{\mu_0}, \dots, x_{\mu_{i-1}}, x_{\mu_{i+1}}, \dots, x_{\mu_k}\} \in \in I_{(\nu_1^0 \dots \nu_k^0)}$ and $\{x_{\mu_0}, \dots, x_{\mu_{i-2}}, x_{\mu_i}, \dots, x_{\mu_k}\} \in I_{(\nu_1^0 \dots k_k^0)}$ and therefore $x_{\mu_{i-1}}, x_{\mu_i} \in S^{\nu_{i-1}^0}_{x_{\mu_0} \dots x_{\mu_{i-2}} x_{\mu_{i+1}} \dots x_{\mu_k}}$, consequently $x_{\mu_i} \notin g_{x_{\mu_0} \dots x_{\mu_{i-2}} x_{\mu_{i+1}} \dots x_{\mu_k}}(\{x_{\mu_{i-1}}\}) = = f(x_{\mu_0}, \dots, x_{\mu_{i-1}}, x_{\mu_{i+1}}, \dots, x_{\mu_k})$.

We have similarly in the case $i = 0$ that $x_{\mu_0} \notin g_{x_{\mu_2} \dots x_{\mu_k}}(\{x_{\mu_1}\}) = = f(x_{\mu_1}, \dots, x_{\mu_k})$. Thus we have proved that there is a free set S_0 of power $\aleph_{\alpha+1}$. Q. e. d.

(*) THEOREM 4. *The smallest m for which the symbols $(m, \aleph_\alpha, k) \to \aleph_\beta$ $(0 \leqq \beta \leqq \alpha + 1)$ are true is $\aleph_{\alpha+k}$.*

Theorem 4 is an immediate consequence of Theorem 3 and Lemma 2.

(*) THEOREM 5. $(\aleph_{\alpha+k}, \aleph_\alpha, k) \not\to \aleph_{\alpha+k}$ *if* $k \geqq 2$; $(\aleph_{\alpha+1}, 2, 2) \not\to \aleph_{\alpha+1}$.

We shall only prove the second statement, the first is a consequence of it. We have stated the first one explicitly, to make Problem 2 clear.

PROOF. Let S be a set, $\overline{\overline{S}} = \aleph_{\alpha+1}$. We define a set-mapping of S of type and order 2 which has no free set of power $\aleph_{\alpha+1}$. Using (*), we have $\overline{\overline{[S]}}^{\aleph_\alpha} = \aleph_{\alpha+1}^{\aleph_\alpha} = \aleph_{\alpha+1}$. Let $\{x_\nu\}_{\nu < \omega_{\alpha+1}}$ and $\{X_\nu\}_{\nu < \omega_{\alpha+1}}$ be well-orderings of type $\omega_{\alpha+1}$ of the sets S and $[S]^{\aleph_\alpha}$, respectively. We define the sets $S_\nu = = \{x_\mu\}_{\mu < \nu}$ and $[S]_\nu^{\aleph_\alpha} = \{X_\mu : X_\mu \subseteq S_\nu;\ \mu < \nu.\}$ Then $\overline{\overline{S}}_\nu \leqq \aleph_\alpha$ and $\overline{\overline{[S]}}_\nu^{\aleph_\alpha} \leqq \aleph_\alpha$ for every $\nu < \omega_{\alpha+1}$. Let $\{x_\iota^\nu\}_{\iota < \omega_\alpha}$ and $\{X_\iota^\nu\}_{\iota < \omega_\alpha}$ be well-orderings of type ω_α of the sets S_ν and $[S]_\nu^\alpha$, respectively. We can choose a sequence $\{\tau_\sigma\}_{\sigma < \omega_\alpha}$ $(\tau_{\sigma_1} < \tau_{\sigma_2}$ if $\sigma_1 < \sigma_2)$ in such a manner that $x_{\tau_\sigma}^\nu \in X_\sigma^\nu$. This sequence may be defined by induction. Suppose that we have already defined $x_{\tau_{\sigma'}}^\nu$, for every $\sigma' < \sigma$, then the set $\{x_{\tau_{\sigma'}}^\nu\}_{\sigma' < \sigma}$ has a power less than $\aleph_\alpha$ and the set X_σ^ν, being an element of $[S]^{\aleph_\alpha}$, has an element x_τ^ν different from all $x_{\tau_{\sigma'}}^\nu$ $(\sigma' < \sigma)$. Let τ_σ be the smallest τ for which $x_\tau^\nu \in X_\sigma^\nu$ and $x_\tau^\nu \notin \{x_{\tau_{\sigma'}}^\nu\}_{\sigma' < \sigma}$.

We define the function $g(x_\mu, x_\nu)$ for $\mu < \nu < \omega_{\alpha+1}$ as follows: We define $g(x_\tau^\nu, x_\nu)$ for every fixed $\nu < \omega_{\alpha+1}$ and for all $\tau < \omega_\alpha$. If τ is an element of the set $\{\tau_\sigma\}_{\sigma < \omega_\alpha}$, we put $g(x_\tau^\nu, x_\nu) = x_{\mu_0}$ where μ_0 denotes the smallest ordinal number μ for which $x_\mu \in X_\sigma^\nu$ $(\tau = \tau_\sigma, x_\mu \neq x_\nu, x_\mu \neq x_{\tau_\sigma}^\nu)$ and put $g(x_\tau^\nu, x_\nu) = x_0$ in the other cases.

Thus we have defined $g(x_\mu, x_\nu)$ for every $\mu < \nu < \omega_{\alpha+1}$. We define the set-mapping $f(X)$ of the set S of type and order 2 as follows:

If $\{x_\mu, x_\nu\}$ $(\mu < \nu)$ is an arbitrary element of $[S]^2$, we put $f(\{x_\mu, x_\nu\}) = = \{g(x_\mu, x_\nu)\}$.

We have to prove that there is no free set of power $\aleph_{\alpha+1}$. Let S_0 be an arbitrary subset of power $\aleph_{\alpha+1}$ of S and S_0' a subset of S_0 such that $\overline{\overline{S_0'}}=\aleph_\alpha$. Then there is a $\nu<\omega_{\alpha+1}$ for which $S_0'=X_\nu$ and a $\nu_0>\nu$ such that $X_\nu\in[S]_{\nu_0}^{\aleph_\alpha}$ and $x_{\nu_0}\in S_0$. Then, by the construction of the function $g(x_\mu, x_\nu)$, there exists a $\mu_0<\nu_0$ for which $x_{\mu_0}\in X_\nu$ and $g(x_{\mu_0}, x_{\nu_0})\in X_\nu$. This means that $\{x_{\mu_0}, x_{\nu_0}\}\subset S_0$ and $f(\{x_{\mu_0}, x_{\nu_0}\})\cdot S_0=\{g(\{x_{\mu_0}, x_{\nu_0}\})\}\neq 0$, consequently the set S_0 is not free. Q. e. d.

(*) THEOREM 6. *If α is an ordinal number of the second kind and $\aleph_\alpha$ is not inaccessible, then $(\aleph_\alpha, \aleph_\beta, k)\to\aleph_\alpha$ for every $\beta<\alpha$ $(k=1,2,\ldots)$.*

PROOF. We prove the theorem by induction on k. For $k=1$ the theorem is well known.[10]

We shall now prove the theorem for $k=2$. Our proof shows clearly how induction works in the general case.

Let S be a set of power $\aleph_\alpha$, $f(X)$ a set-mapping of S of type 2 and order $\aleph_\beta$. Let τ be the smallest ordinal number for which $\aleph_\alpha=\sum_{\nu<\tau}\aleph_{\beta_\nu}$ $(\beta_\mu<\beta_\nu<\alpha$ if $\mu<\nu)$. By the assumption of the theorem $\overline{\overline{\tau}}<\aleph_\alpha$. Since $\beta<\alpha$, we may suppose that $\beta+2<\beta_\nu$ for every $\nu<\tau$.

We define $\aleph_{\gamma_\nu}$ as the sum $\sum_{\mu<\nu}\aleph_{\beta_\mu}$. We may also suppose that $\aleph_{\beta_\nu}\geqq\aleph_{\gamma_\nu+3}$ and that every β_ν is of the form $\gamma+s$ where $s\geqq 3$.

By Theorem 3 every subset of power $\aleph_\alpha$ of the set S contains a free subset of power $\aleph_{\beta_\nu}$. So we may select a sequence $\{S_\nu\}_{\nu<\tau}$ of the subsets of S which satisfies the following conditions:

(1) $\overline{\overline{S_\nu}}=\aleph_{\beta_\nu}$; (2) S_ν is a free set; (3) $S_{\nu_1}\cdot S_{\nu_2}=0$ if $\nu_1\neq\nu_2$.

Put $F_\nu=\sum_{\mu<\nu}S_\mu$. By (1) and (3) $\overline{\overline{F_\nu}}=\aleph_{\gamma_\nu}$. Thus $\overline{\overline{[F_\nu]^2}}=\aleph_{\gamma_\nu}$, consequently $\overline{\overline{\sum_{X\in[F_\nu]^2}f(X)}}\leqq\aleph_{\gamma_\nu}\cdot\aleph_\beta=\aleph_{\gamma_\nu}<\aleph_{\beta_\nu}$. Therefore we may suppose that

$$f(X)\cdot S_\nu=0 \quad \text{if} \quad X\in[F_\nu]^2. \tag{4}$$

Let S^0 denote the set $\sum_{\nu<\tau}S_\nu$. If $X\in[S^0]^2$, then $X=\{x,y\}$. In what follows $f(X)\cdot S^0$ will be denoted by $f_1(x,y)$. Suppose now that $x,y\in S_\nu$. Then $f_1(x,y)\cdot S_\mu=0$ for $\mu>\nu$ by (4) and for $\mu=\nu$ by (2). Thus $f_1(x,y)\subseteq F_\nu$ and since $\overline{\overline{f_1(x,y)}}<\aleph_\beta$, $f_1(x,y)\in[F_\nu]^{<\aleph_\beta}$.

We split the set $[S_\nu]^2$ into classes. The sets $\{x_1,y_1\}$ and $\{x_2,y_2\}$ belong to the same class if and only if $f_1(x_1,y_1)=f_1(x_2,y_2)$. Using (*), we have $\overline{\overline{[F_\nu]^{<\aleph_\beta}}}\leqq\aleph_{\gamma_\nu}^{\aleph_\beta}\leqq\aleph_{\gamma_\nu+1}\leqq\aleph_{\beta_\nu-2}$. Thus, by Lemma 3, there is an $S_\nu'\subseteq S_\nu$

[10] See e. g. [1].

($\overline{\overline{S'_\nu}} \geqq \aleph_{\beta_\nu-1}$) such that for every $x, y \in S'_\nu$ $f_1(x,y)$ is the same subset of F_ν. We define S^1_ν as follows: $S^1_\nu = S'_\nu - \sum_{\mu > \nu,\ \{x,y\} \subset S'_\mu} f_1(x,y)$. It is obvious that

$$\overline{\overline{\sum_{\mu>\nu,\ \{x,y\}\subset S'_\mu} f_1(x,y)}} \leqq \overline{\tau}\cdot\aleph_\beta \leqq \aleph_{\beta_\nu-2} < \aleph_{\beta_\nu-1}$$

and therefore we have

$$\overline{\overline{S}}{}^1_\nu \geqq \aleph_{\beta_\nu-1}. \tag{5}$$

Let S^1 be the set $\sum_{\nu<\tau} S^1_\nu$. By the construction of S^1_ν we have

$$f_1(x,y)\cdot S^1 = 0 \quad \text{if} \quad x, y \in S^1_\nu \quad \text{for every} \quad \nu < \tau. \tag{6}$$

Let F^1_ν be the set $\sum_{\mu<\nu} S^1_\mu$. We define the set-mapping $g_\nu(\{y\})$ of the set S^1_ν of type 1 as follows: For every element y of S^1_ν let $g_\nu(\{y\}) = \sum_{x\in F^1_\nu} f_1(x,y)\cdot S^1_\nu$.

We have

$$\overline{\overline{\sum_{x\in F^1_\nu} f_1(x,y)}} \leqq \overline{\overline{F}}{}^1_\nu\cdot\aleph_\beta \leqq \aleph_{\gamma_\nu}\cdot\aleph_\beta = \aleph_{\gamma_\nu}.$$

Thus the order of the set-mapping $g_\nu(\{y\})$ is less than $\aleph_{\beta_\nu-1}$. By the case $k=1$ of Theorem 3 there is an $S^2_\nu \subseteq S^1_\nu$ ($\overline{\overline{S}}{}^2_\nu = \aleph_{\beta_\nu-1}$) such that S^2_ν is a free set of $g_\nu(\{y\})$.

Let S^2 be the set $\sum_{\nu<\tau} S^2_\nu$ and F^2_ν the set $\sum_{\mu<\nu} S^2_\mu$. We have by the construction of S^2_ν

$$\overline{\overline{S}}{}^2_\nu = \aleph_{\beta_\nu-1}. \tag{7}$$

From (6) we have

$$f_1(x,y)\cdot S^2 = 0 \quad \text{if} \quad x, y \in S^2_\nu \quad \text{for every} \quad \nu < \tau, \tag{8}$$

and from (4), by the construction of S^2_ν, we get

$$f_1(x,y)\cdot \sum_{\mu\geqq\nu} S^2_\mu = 0 \quad \text{if} \quad x \in F^2_\nu \quad \text{and} \quad y \in F^2_\nu \quad \text{or} \quad y \in S^2_\nu. \tag{9}$$

Put $f_2(x,y) = f_1(x,y)\cdot S^2$.

If $x \in F^2_\nu$ and $y \in S^2_\nu$, then, by (9), $f_2(x,y) \subseteq F^2_\nu$. We split the set S^2_ν into classes as follows: y and z belong to the same class if and only if $f_2(x,y) = f_2(x,z)$ for every $x \in F^2_\nu$.

We can obtain these classes as follows: First we split S^2_ν into classes, corresponding to every $x_0 \in F^2_\nu$, so that y and z belong to the same class if $f_2(x_0,y) = f_2(x_0,z)$ for this x_0 and thus we obtain at most $\overline{\overline{[F^2_\nu]^{<\aleph_\beta}}} \leqq \aleph_{\gamma_\nu}^{\aleph_\beta} \leqq \aleph_{\gamma_\nu+1}$ classes, since $f_2(x,y) \in [F^2_\nu]^{<\aleph_\beta}$. y and z belong to the same class if they belong to the same class for all x_0. Therefore we have split the set S_ν^2 into at most $\aleph_{\gamma_\nu+1}^{\aleph_{\gamma_\nu}}=\aleph_{\gamma_\nu+1}$ classes, consequently, by (7), there is a subset S_ν^3 of power $\aleph_{\beta_\nu-1}$ of S_ν^2 whose all elements belong to the same class. It follows that

$$\overline{\overline{S}}_\nu^3=\aleph_{\beta_\nu-1}. \tag{10}$$

Let S^3 be the set $\sum_{\nu<\tau} S_\nu^3$ and F_ν^3 the set $\sum_{\mu<\nu} S_\mu^3$. If

$$x\in F_\nu^3,\quad y,z\in S_\nu^3, \tag{11}$$

then, by the construction of S_ν^3, $f_2(x,y)=f_2(x,z)$. Further, by (8),

$$f_2(x,y)\cdot S^3=0 \quad\text{for}\quad x,y\in S_\nu^3. \tag{12}$$

We define the set-mapping $g_0(\{x\})$ of the set S^3 of type 1 as follows:

For every $x\in S^3$ there is exactly one $\nu<\tau$ for which $x\in S_\nu^3$. Put $g_0(\{x\})=\sum_{\mu>\nu,\, y\in S_\mu^3} f_2(x,y)\cdot S^3$ for all $x\in S^3$ where ν is the ordinal number for which $x\in S_\nu^3$. It follows from (11) that $\overline{\overline{g_0(\{x\})}}\leqq\overline{\overline{\tau}}\cdot\aleph_\beta<\aleph_\alpha$.

The set S^3 has the power $\aleph_\alpha$. This is true because by (3) the sets S_ν^3 are mutually exclusive and by (10) $\overline{\overline{S}}_\nu^3=\aleph_{\beta_\nu-1}$. Thus, by the induction hypothesis, i. e. by the case $k=1$ of our theorem which is already proved, there is a free set S^4 of $g_0(\{x\})$ such that $S^4\subseteq S^3$ and $\overline{\overline{S}}^4=\aleph_\alpha$. Let S_ν^4 be the set $S^4\cdot S_\nu^3$. Then $S^4=\sum_{\nu<\tau} S_\nu^4$. If $x,y\in S^4$, then there is a μ and ν such that $x\in S_\mu^4$ and $y\in S_\nu^4$. We may suppose that $\mu\leqq\nu$. Obviously, $f(\{x,y\})\cdot S^4=f_2(x,y)\cdot S^4$ and, by (12), $f_2(x,y)\cdot S^4=0$ if $\mu=\nu$. Further $f_2(x,y)\subseteq g_0(\{x\})$ if $\mu<\nu$. Therefore $f_2(x,y)\cdot S^4=0$ in this case too, since S^4 is a free set of $g_0(\{x\})$. Thus S^4 is a free set of power $\aleph_\alpha$ of the set-mapping $f(x)$. Q. e. d.

In the proof of Theorem 6 we have made use of the assumption that $\aleph_\alpha$ is not inaccessible. In the case when $\aleph_\alpha$ is inaccessible, we can prove $(\aleph_\alpha,\aleph_\beta,k)\to\aleph_\alpha$ $(\beta<\alpha)$ only if we use (**). But using (**), we can prove the following much stronger result:

(**) THEOREM 7. *If the cardinal number $\aleph_\alpha>\aleph_0$ is strongly inaccessible, then $(\aleph_\alpha,\aleph_\beta,\omega)\to\aleph_\alpha$ for every $\beta<\alpha$.*

PROOF. Let S be a set, $\overline{\overline{S}}=\aleph_\alpha$ and $f(X)$ a set-mapping of S of type ω and order $\aleph_\beta$. We have to prove the existence of a free set of power $\aleph_\alpha$.

Let $f(x_1,\dots,x_k)$ denote briefly $f(\{x_1,\dots,x_k\})$. Let $\mu(X)$ be the two-valued measure defined on all subsets of S the existence of which is assured by the hypothesis (**).

Let $\{x_1, \ldots, x_k\}$ be an arbitrary element of $[S]^k$ and $y \in S$. We define the set $S^y_{x_1 \ldots x_k}$ as follows: Let

(1) $$S^y_{x_1 \ldots x_k} = \{x : y \in f(x_1, \ldots, x_k, x);\ x \neq x_1, \ldots,\ x \neq x_k\}.$$

We define the set-mapping $f'(x_1, \ldots, x_k)$ of the set S of type ω as follows: Let

(2) $$f'(x_1, \ldots, x_k) = \{y : \mu(S^y_{x_1 \ldots x_k}) = 1\} \quad \text{for every} \quad \{x_1, \ldots, x_k\} \in [S]^k.$$

We call f' the derived set-mapping of f. First we prove the following lemma:

(3) *If f is an arbitrary set-mapping of S of type ω and order $\aleph_\beta$ $(\beta < \alpha)$, then the derived set-mapping f' is of order $\aleph_\beta$ too.*

To see this assume $\overline{\overline{f'(x_1, \ldots, x_k)}} \geqq \aleph_\beta$ for a certain $\{x_1, \ldots, x_k\} \in [S]^k$. Then there is a set Y $(\overline{\overline{Y}} = \aleph_\beta)$ such that $Y \subseteq f'(x_1, \ldots, x_k)$. Let $\{y_\nu\}_{\nu < \omega_\beta}$ be a well-ordering of Y of type ω_β. For every $\nu < \omega_\beta$, by (2), $\mu(S^{y_\nu}_{x_1 \ldots x_k}) = 1$ and therefore $\mu(\prod_{\nu < \omega_\beta} S^{y_\nu}_{x_1 \ldots x_k}) = 1$, since $\beta < \alpha$ and the measure μ is additive for less than $\aleph_\alpha$ summands. Thus the set $\prod_{\nu < \beta} S^{y_\nu}_{x_1 \ldots x_k}$ is non-empty; let x_0 be an element of it. Then $y_\nu \in f(x_1, \ldots, x_k, x_0)$ for every $\nu < \omega_\beta$, therefore $Y \subseteq f(x_1, \ldots, x_k, x_0)$, consequently $\overline{\overline{f(x_1, \ldots, x_k, x_0)}} \geqq \aleph_\beta$. This is a contradiction, because f is of order $\aleph_\beta$. Therefore $\overline{\overline{f'(x_1, \ldots, x_k)}} < \aleph_\beta$ for every $\{x_1, \ldots, x_k\} \in [S]^k$.

Now we define the set-mappings $f_l(X)$ of S by induction on l.

Put $f_0(x) = f(x)$ and $f_{l+1}(x) = f_l'(x)$. We define $S^{y,l}_{x_1 \ldots x_k}$ writing $f_l(X)$ instead of $f(X)$ in (1).

(4) The set-mappings $f_l(X)$ of type ω are, by (3), of order $\aleph_\beta$.

Now we define a sequence $\{x_\nu\}_{\nu < \omega_\alpha}$ by induction as follows: Let x_0 be an arbitrary element of S. Suppose that we have already defined the elements x_μ for all $\mu < \nu$, where $\nu < \omega_\alpha$ is a given ordinai number, such that $x_\mu \in S$.

Let S^ν be the set $\{x_\mu\}_{\mu < \nu}$ and S^ν_1 the set $\sum_{X \in [S^\nu]^{<\aleph_0};\, l = 0, 1, 2, \ldots} f_l(X)$. We define the sets $\bar{S}^{y,l}_{x_1 \ldots x_k}$ as follows: Let

(5) $$\bar{S}^{y,l}_{x_1 \ldots x_k} = \begin{cases} S^{y,l}_{x_1 \ldots x_k} & \text{if} \quad \mu(S^{y,l}_{x_1 \ldots x_k}) = 0, \\ 0 & \text{if} \quad \mu(S^{y,l}_{x_1 \ldots x_k}) = 1. \end{cases}$$

Let further S^ν_2 be the set $\sum_{y, x_1, \ldots, x_k \in S^\nu;\, l = 0, 1, 2, \ldots} \bar{S}^{y,l}_{x_1 \ldots x_k}$. It follows that $\overline{\overline{S^\nu}} \leqq \bar{\nu} < \aleph_\alpha$ and $\overline{\overline{S^\nu_1}} \leqq \bar{\nu} \cdot \aleph_0 \cdot \aleph_\beta < \aleph_\alpha$. From (5) it follows by the additivity of the measure that $\mu(S^\nu_2) = 0$. Thus the set $S - (S^\nu + S^\nu_1 + S^\nu_2)$ is of measure 1 and therefore it is non-empty. Let x_ν be an arbitrary element of it.

Thus we have defined x_ν for all $\nu < \omega_\alpha$ and it is obvious from the construction that $x_\nu \neq x_\mu$ for $\nu \neq \mu$. Therefore the set $S_1 = \{x_\nu\}_{\nu < \omega_\alpha}$ is of power $\aleph_\alpha$.

We define the set-mapping $g(\{x\})$ of S_1 of type 1. Let $g(\{x\}) = \sum_{l=0}^{\infty} f_l(x) \cdot S_1$ for every $x \in S_1$. We have by (4) that $\overline{\overline{g(\{x\})}} \leqq \aleph_\beta \cdot \aleph_0$, i. e. $g(\{x\})$ is of order $\aleph_{\beta+1} < \aleph_\alpha$. Then, by a theorem of ERDŐS already cited,[11] there is a free set $S_2 \subseteq S_1$ of power $\aleph_\alpha$ of the set-mapping $g(\{x\})$.

We shall prove that S_2 is a free set of the set-mapping $f(x)$ too. Since $S_2 \subseteq S_1$, S_2 has the form $\{x_{\nu_\mu}\}_{\mu < \omega_\alpha}$. If S_2 is not a free set of $f(X)$, then there is a subset $\{x_{\nu_{\mu_0}}, \dots, x_{\nu_{\mu_k}}\}$ $(\mu_0 < \dots < \mu_k)$ of S_2 for which $x_{\nu_{\mu_i}} \in f(x_{\nu_{\mu_0}}, \dots, x_{\nu_{\mu_{i-1}}}, x_{\nu_{\mu_{i+1}}}, \dots, x_{\nu_{\mu_k}})$ for an i $(0 \leqq i \leqq k)$. If $i = k$, then $x_{\nu_{\mu_k}} \in S_1^{\nu_{\mu_k}}$, in contradiction to the construction. Therefore we may suppose that $i < k$. But if $i < k$, then by the construction $x_{\nu_{\mu_k}} \notin S_2^{\nu_{\mu_k}}$. Thus $\mu(S^{x_{\nu_{\mu_i}}, 0}_{x_{\nu_{\mu_1}} \dots x_{\nu_{\mu_{i-1}}} x_{\nu_{\mu_{i+1}}} \dots x_{\nu_{\mu_{k-1}}}}) = 1$, i. e., by (2),

$$x_{\nu_{\mu_i}} \in f_1(x_{\nu_{\mu_0}}, \dots, x_{\nu_{\mu_{i-1}}}, x_{\nu_{\mu_{i+1}}}, \dots, x_{\nu_{\mu_{k-1}}}).$$

Repeating these considerations for $f_1, f_2, \dots$ instead of f_0, we obtain that $i = 0$ and

$$x_{\nu_{\mu_0}} \in f_{k-1}(x_{\nu_{\mu_1}}).$$

But this is a contradiction, because S_2 is a free set of the set-mapping $g(\{x\})$ and $f_{k-1}(x_{\nu_{\mu_1}}) \subseteq g(\{x_{\nu_{\mu_1}}\})$. Thus S_2 must be a free set of $f(x)$ and $\overline{\overline{S_2}} = \aleph_\alpha$. Q. e. d.

(**) THEOREM 8. *If α is an ordinal number of the second kind and $\beta < \alpha$, then $(\aleph_\alpha, \aleph_\beta, k) \to \aleph_\alpha$ for $k = 1, 2, \dots$.*

Theorem 8 is a consequence of Theorems 6 and 7.

THEOREM 9.[12]

9a) *Let m_0 be a strongly inaccessible cardinal number, $m_0 > \aleph_0$, $\overline{\overline{S}} = m_0$ and $[S]^k = I_1^k + I_2^k$ for $k = 1, 2, \dots$. Then there exists a subset $S_0 \subseteq S$ and a series $\{n_k\}_{k=1,2,\dots}$, where $n_k = 1, 2$, such that $\overline{\overline{S_0}} = m_0$ and $[S]^k \subseteq I_{n_k}^k$ for every k.*

9b) *Let m_0 be the first strongly inaccessible cardinal number greater than $\aleph_0$ and $m < m_0$. Let S be a set, $\overline{\overline{S}} = m$. One can define the classes I_1^k, I_2^k for every k so that the following conditions hold:*

[11] See [1].

[12] Theorem 9 gives the solution of the problem of P. ERDŐS and R. RADO mentioned on p. 113. The statement 9b) was first proved by G. FODOR.

(1) $I_1^k \cdot I_2^k = 0$ for $k = 1, 2, \ldots$;

(2) $[S]^k = I_1^k + I_2^k$ for $k = 1, 2, \ldots$;

(3) *for every infinite subset* S_0 *of* S *there is a* k *such that neither* $[S_0]^k \subseteq I_1^k$ *nor* $[S_0]^k \subseteq I_2^k$.

PROOF. The idea of the proof of 9a) is the same as the one we have used to prove Theorem 8. Therefore we shall only sketch this proof.

We define the classes $I_1^{k,l}, I_2^{k,l}$ for every $l < \omega$ by induction on l.

Put $I_1^{k,0} = I_1^k$, $I_2^{k,0} = I_2^k$. Suppose that $I_1^{k,l}$, $I_2^{k,l}$ are already defined and let $\{x_1, \ldots, x_k\}$ be an arbitrary element of $[S]^k$. Put $S^{1,l}_{x_1 \ldots x_k} = \{x : \{x_1, \ldots, x_k, x\} \in I_1^{k+1,l}\}$ and $S^{2,l}_{x_1 \ldots x_k} = \{x : \{x_1, \ldots, x_k, x\} \in I_2^{k+1,l}\}$. We shall say that $\{x_1, \ldots, x_k\} \in I_1^{k,l+1}$ if $\mu(S^{1,l}_{x_1 \ldots x_k}) = 1$ and $\{x_1, \ldots, x_k\} \in I_2^{k,l+1}$ if $\mu(S^{2,l}_{x_1 \ldots x_k}) = 1$. Let ω_α be the initial number of m_0. We define the sequence $\{x_\nu\}_{\nu < \omega_\alpha}$ by induction. Let x_0 be an arbitrary element of S. Suppose that x_μ is already defined for all $\mu < \nu$. For every l and X $(X \in [\{x_\mu\}_{\mu < \nu}]^{<\aleph_0})$ there is an $n(X, l)$ $(n(X, l) = 1$ or $n(X, l) = 2)$ such that $\mu(S_X^{n(X,l),l}) = 1$. The set $\prod_{X \in [\{x_\mu\}_{\mu<\nu}]^{<\aleph_0};\, l = 0,1,2,\ldots} S_x^{n(X,l),l}$ is non-empty and we define x_ν as an arbitrary element of it.

Put $S_1 = \{x_\nu\}_{\nu < \omega_\alpha}$. We split the set S_1 into classes, x and y belong to the same class if and only if for every $l < \omega$ $\{x\} \in I_1^{1,l}$ holds if and only if $\{y\} \in I_1^{1,l}$ holds. Thus we obtain at most $2^{\aleph_0}$ classes, and therefore there is a class $S_0 \subseteq S_1$ of power m_0 which satisfies the requirements of 9a). Indeed, S_0 has the form $\{x_{\nu_\mu}\}_{\mu < \omega_\alpha}$. Suppose that for a k neither $[S_0] \subseteq I_1^k$ nor $[S_0]^k \subseteq I_2^k$. Then there is a set $\{x_{\nu_{\mu_1}}, \ldots, x_{\nu_{\mu_k}}\}$ $(\mu_1 < \cdots < \mu_k)$ and a set $\{x_{\nu_{\mu'_1}}, \ldots, x_{\nu_{\mu'_k}}\}$ $(\mu'_1 < \cdots < \mu'_k)$ for which $\{x_{\nu_{\mu_1}}, \ldots, x_{\nu_{\mu_k}}\} \in I_1^{k,0}$ and $\{x_{\nu_{\mu'_1}}, \ldots, x_{\nu_{\mu'_k}}\} \in I_2^{k,0}$. Then, by the construction, $\{x_{\nu_{\mu_1}}, \ldots, x_{\nu_{\mu_{k-1}}}\} \in I_1^{k-1,1}$ and $\{x_{\nu_{\mu'_1}}, \ldots, x_{\nu_{\mu'_{k-1}}}\} \in I_2^{k-1,1}$, ..., and finally $\{x_{\nu_{\mu_1}}\} \in I_1^{1,k-1}$, $\{x_{\nu_{\mu'_1}}\} \in I_2^{1,k-1}$, but this contradicts the fact that $x_{\nu_{\mu_1}}$, $x_{\nu_{\mu'_1}}$ belong to the same class S_0.

To prove 9b) we shall first prove the following:

(i) If the statement is true for $m = \aleph_\alpha$, then it is true for $m = 2^{\aleph_\alpha}$.

Let S_1 be the set $\{\nu\}_{\nu < \omega_\alpha}$. Then $\overline{\overline{S}}_1 = \aleph_\alpha$ and by the assumption of (i) one can define the classes I_1^k, I_2^k satisfying the conditions (1), (2), (3) of 9b) (with S_1 instead of S). To prove (i) it is sufficient to define a set S of power $2^{\aleph_\alpha}$ and the classes I_{I}^k, I_{II}^k satisfying the conditions (1), (2) and (3).

Let S be the set of all sequences $\{\varepsilon_\nu\}_{\nu < \omega_\alpha}$ where $\varepsilon_\nu = 0$ or $\varepsilon_\nu = 1$. Then $\overline{\overline{S}} = 2^{\aleph_\alpha}$. Let x_1, x_2 be two arbitrary elements of S, $x_1 \neq x_2$, $x_1 = \{\varepsilon_\nu^1\}_{\nu < \omega_\alpha}$

and $x_2=\{\varepsilon_\nu^2\}_{\nu<\omega_\alpha}$. Let $\nu(x_1,x_2)$ denote the smallest ordinal number ν for which $\varepsilon_\nu^1 \neq \varepsilon_\nu^2$.

We define the usual lexicografical ordering of S, we say that $x_1<x_2$ if and only if $\varepsilon^1_{\nu(x_1,x_2)}=0$.

Now let $\{x_1,\dots,x_k\}$ be an arbitrary subset of k elements of S. We may suppose that $x_1<\dots<x_k$. We write $\nu_1,\dots,\nu_{k-1}$ instead of $\nu(x_1,x_2),\dots,\nu(x_{k-1},x_k)$.

We define the classes I_{I}^k, I_{II}^k for $k\geqq 2$ as follows:

(a) if $\nu_1<\dots<\nu_{k-1}$ or $\nu_1>\dots>\nu_{k-1}$, then we say that

$$\{x_1,\dots,x_k\}\in\begin{cases} I_{\mathrm{I}}^k & \text{if } \{\nu_1,\dots,\nu_{k-1}\}\in I_1^{k-1},\\ I_{\mathrm{II}}^k & \text{if } \{\nu_1,\dots,\nu_{k-1}\}\in I_2^{k-1};\end{cases}$$

(b) in the other cases let $\{x_1,\dots,x_k\}\in I_{\mathrm{I}}^k$.

Obviously, conditions (1) and (2) hold for S, I_{I}^k and I_{II}^k. We have to prove that condition (3) holds too.

Let S_0 be an infinite subset of S. It is well known that there exists a sequence $\{x_l\}_{l<\omega}\subseteq S_0$ for which either $x_1<\dots<x_l<\cdots$ or $x_1>\dots>x_l>\cdots$.

We shall define a subsequence $\{x_{l_s}\}_{s<\omega}$ for which $\mu_{l_1}<\dots<\mu_{l_s}<\cdots$, where $\mu_{l_s}=\nu(x_{l_s},x_{l_{s+1}})$. Without loss of generality we may assume that $x_1<\dots<x_l<\cdots$. Let ν' denote the smallest ordinal number which occurs among the ordinals $\nu(x_l,x_{l'})$ $(l<\omega, l'<\omega, l\neq l')$. Let l' be the smallest integer for which $\varepsilon_{\nu'}^{l'}=1$ and let l be the greatest integer less than l' for which $\varepsilon_{\nu'}^{l}=0$. Put $x_{l_1}=x_l$. It is obvious that $\nu(x_{l_1},x_{l''})=\nu'$ for every $l''\geqq l'$ and $\nu(x_{l''},x_{l'''})>\nu'$ for $l'',l'''\geqq l'$. If we repeat this for the sequence $x_{l'},x_{l'+1},\dots$ we obtain an element x_{l_2} and so on. The sequence $\{x_{l_s}\}_{s<\omega}$ satisfies our requirement.

In what follows we write x_s instead of x_{l_s} and μ_s instead of μ_{l_s}. Let S' denote the a set $\{x_s\}_{s<\omega}$ and S_1' the set $\{\mu_s\}_{s<\omega}$.

If $\{\mu_{s_1},\dots,\mu_{s_{k-1}}\}$ $(\mu_{s_1}<\dots<\mu_{s_{k-1}})$ is an arbitrary finite subset of S_1', then $\{x_{s_1},\dots,x_{s_{k-1}},x_{s_k}\}$ with $s_k=s_{k-1}+1$ is a subset of k elements of S' such that $\nu(x_{s_1},x_{s_2})=\mu_{s_1},\dots,\nu(x_{s_{k-1}},x_{s_k})=\mu_{s_{k-1}}$. This means by (a) that to every finite subset of $k-1$ elements of S_1' there is a subset of k elements of S' such that the second belongs to I_{I}^k or to I_{II}^k if and only if the first belongs to I_1^k or to I_2^k, respectively. But there is a k for which neither $[S_1']^k\subseteq I_1^k$ nor $[S_1']^k\subseteq I_2^k$ and for this k neither $[S']^{k+1}\subseteq I_{\mathrm{I}}^{k+1}$ nor $[S']^{k+1}\subseteq I_{\mathrm{II}}^{k+1}$. Thus the sets S, I_{I}^k, I_{II}^k satisfy condition (3) too, and so (i) is proved.

Let ω_{α_0} denote the initial number of m_0.

(ii) If α is an ordinal number of the second kind, $\alpha<\alpha_0$ and if the statement of 9b) is true for every $m=\aleph_\beta$ $(\beta<\alpha)$, then it is true for $m=\aleph_\alpha$.

Proof of (ii). If $\aleph_\alpha$ is inaccessible, then it can not be strongly inaccessible, because $0 < \alpha < \alpha_0$. Thus if $\aleph_\alpha$ is inaccessible, then there is a $\beta < \alpha$ for which $\aleph_\alpha \leqq 2^{\aleph_\beta}$, but then, by (i), 9b) is true for $\aleph_\alpha$. Thus we may suppose that $\aleph_\alpha$ is not inaccessible.

Let S be a set, $\overline{\overline{S}} = \aleph_\alpha$ and τ the smallest ordinal number for which $\aleph_\alpha = \sum_{\nu<\tau} \aleph_{\beta_\nu}$, $\beta_{\nu_1} < \beta_{\nu_2} < \alpha$ for $\nu_1 < \nu_2 < \tau$. In this case we have $\overline{\overline{\tau}} < \aleph_\alpha$. There is a sequence $\{S_\nu\}_{\nu<\tau}$ for which $S = \sum_{\nu<\tau} S_\nu$, $S_{\nu_1} \cdot S_{\nu_2} = 0$ if $\nu_1 \neq \nu_2$ and $\overline{\overline{S}}_\nu = \aleph_{\beta_\nu}$. By the assumption corresponding to every set S_ν $(\nu < \tau)$, we can define the sets $I_1^{k,\nu}$, $I_2^{k,\nu}$ so that (1), (2) and (3) hold for $S_\nu, I_1^{k,\nu}, I_2^{k,\nu}$ instead of S, I_1^k, I_2^k, respectively. Put $S^* = \{\nu\}_{\nu<\tau}$. Then $\overline{\overline{S^*}} < \aleph_\alpha$ and we can define the sets $I_1^{k,*}$, $I_2^{k,*}$ satisfying the conditions (1), (2) and (3).

Now we define the sets I_1^k, I_2^k for $k = 1, 2, \ldots$ as follows: Let $\{x_1, \ldots, x_k\}$ be an arbitrary element of $[S]^k$.

a) If for every i $(1 \leqq i \leqq k)$ $x_i \in S_{\nu_i}$ and $\nu_i \neq \nu_j$ for every $i \neq j$, then let

$$\{x_1, \ldots, x_k\} \in \begin{cases} I_1^k & \text{if } \{\nu_1, \ldots, \nu_k\} \in I_1^{k,*}, \\ I_2^k & \text{if } \{\nu_1, \ldots, \nu_k\} \in I_2^{k,*}. \end{cases}$$

b) If there is a ν for which $\{x_1, \ldots, x_k\} \subseteq S_\nu$, then let

$$\{x_1, \ldots, x_k\} \in \begin{cases} I_1^k & \text{if } \{x_1, \ldots, x_k\} \in I_1^{k,\nu}, \\ I_2^k & \text{if } \{x_1, \ldots, x_k\} \in I_2^{k,\nu}. \end{cases}$$

c) Let $\{x_1, \ldots, x_k\} \in I_1^k$ in the other cases.

It is obvious that conditions (1) and (2) hold for I_1^k, I_2^k.

Let S' be an arbitrary infinite subset of S. Then either (o) $S' \cdot S_\nu \neq 0$ for infinitely many ν or (oo) there is a ν such that $\overline{\overline{S_\nu \cdot S'}} \geqq \aleph_0$.

If (o) holds, then there is a subset S'' of S' such that S'' is infinite and $\overline{\overline{S'' \cdot S_\nu}} = 1$ for every ν. Then, by the case a), there is a k for which neither $[S'']^k \subseteq I_1^k$ nor $[S'']^k \subseteq I_2^k$.

If (oo) holds, then there is a ν and a subset S''' of S' such that $S''' \subseteq S_\nu$ and $\overline{\overline{S'''}} \geqq \aleph_0$. Then, by the case b), there is a k such that neither $[S''']^k \subseteq I_1^k$ nor $[S''']^k \subseteq I_2^k$.

It follows that condition (3) holds, consequently (ii) is proved.

The statement of 9b) is true for $m = \aleph_0$.[13]

It follows from (i) and (ii) by transfinite induction that 9b) is true for every $m = \aleph_\alpha$ where $\alpha < \alpha_0$.

REMARK. In the proof of 9b) neither (**) nor (*) are used.

[13] See e.g. [5].

LEMMA 5. *Let S be a set, $\overline{\overline{S}} \geqq \aleph_0$ and $[S]^{k+1} = I_0 + \cdots + I_l$. Then there is an i $(0 \leqq i \leqq l)$ and a subset $S_0 \subseteq S$ such that $\overline{\overline{S_0}} \geqq \aleph_0$ and $[S_0]^{k+1} \subseteq I_i$ for $k = 0, 1, 2, \ldots$; $l = 0, 1, 2, \ldots$.*

Lemma 5 is a theorem of RAMSEY.[14]

THEOREM 10. *$(m, l+1, k) \to \aleph_0$ if $m \geqq \aleph_0$ for $l = 1, 2, \ldots$; $k = 1, 2, \ldots$.*

PROOF. Let S be a set, $\overline{\overline{S}} \geqq \aleph_0$ and $f(X)$ a set-mapping of S of type k and order $l+1$. We shall prove the existence of a free set of power $\aleph_0$.

For every $X \in [S]^k$ the set $f[X]$ has at most l elements. Let $f_1(X), \ldots, f_l(X)$ denote the elements of the set $f(X)$. (If $f(X)$ has less than l elements, then one element may occur more than once.)

Let us split the set $[S]^{k+1}$ into the sum of the sets $I_0, \ldots, I_l$ as follows: If $\{x_0, \ldots, x_k\}$ is an arbitrary element of $[S]^{k+1}$, then it is an element of I_i (for $i = 1, \ldots, l$) if and only if there is a j $(0 \leqq j \leqq k)$ for which $x_j = f_i(\{x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k\})$. If such a j does not exist, then $\{x_0, \ldots, x_k\}$ belongs to I_0.

Obviously $[S]^{k+1} = I_0 + \cdots + I_k$.

Let now S'' be a finite subset of S and suppose that it has s elements. If $[S'']^{k+1} \subseteq I_i$ for some i $(1 \leqq i \leqq l)$, then the set of subsets of S'' taken $k+1$ at a time has at most as many elements as the set of subsets of S'' taken k at a time. It follows that $\binom{s}{k+1} \leqq \binom{s}{k}$, i. e. $s \leqq 2k+1$. Therefore, for every infinite subset S' of S, $[S']^{k+1} \subseteq\!\!\!\!/\; I_i$ for $i = 1, \ldots, k$. Then, by Lemma 5, there is an infinite subset S_0 of S for which $[S_0]^{k+1} \subseteq I_0$, but then S_0 is obviously free. Q. e. d.

(**) THEOREM 11. *$(\aleph_{\alpha+k-1}, l+1, k) \to \aleph_\alpha$ if α is of the first kind; $(\aleph_\alpha, l+1, k) \to \aleph_\alpha$ if α is of the second kind $(l = 1, 2, \ldots; k = 1, 2, \ldots)$.*

The first part of the theorem is a consequence of Theorem 3. The second one follows for $\alpha > 0$ from Theorem 8, and for $\alpha = 0$ from Theorem 10. The hypothesis (**) is used only in the case when $\aleph_\alpha$ is inaccessible.

We have stated Theorem 11 explicitly only to make Problem 3 clear.

In what follows let p, m, l and k denote integers.

THEOREM 12. *Let $p(m, l, k)$ denote the greatest integer p for which $(m, l+1, k) \to p$ is true. Then*

$$c_1 \sqrt[k+1]{m} < p(m, l, k) < c_2 \sqrt[k]{m \log m}$$

where the numbers c_1 and c_2 depend on k and l but they does not depend on m, and $c_1 > 0$.

[14] See [9].

PROOF. First we prove (i) $c_1\sqrt[k+1]{m} < p(m, l, k)$.

Let S be a set, $\overline{\overline{S}} = m$ and $f(X)$ a set-mapping of S of type k and order $l+1$.

If p is an integer for which there is no free subset of power $\geqq p$, then every $X \in [S]^p$ has a subset $Y \in [S]^{k+1}$ such that Y is not free. But if $Y \in [S]^{k+1}$, then there exist in $[S]^p$ at most $\binom{m-k-1}{p-k-1}$ sets X such that $Y \subseteq X$. Therefore there are at least $\binom{m}{p} \Big/ \binom{m-k-1}{p-k-1}$ sets Y in $[S]^{k+1}$ which are not free. On the other hand, we know that there are at most $\binom{m}{k} l$ such Y's and therefore we have

$$\frac{\binom{m}{p}}{\binom{m-k-p}{p-k-1}} < l\binom{m}{k}.$$

It follows that for such a p

$$c\sqrt[k+1]{m} \leqq p$$

for some $c > 0$ which proves (*i*).

Now we outline the proof of (ii) $p(m, l, k) < c_2\sqrt[k]{m \log m}$. Let S be a set, $\overline{\overline{S}} = m$ and M the set of all set-mappings of S of type k and order $l+1$. We define on M a probability field such that every element f of M has the same probability. Let A be the following event: The set-mapping $f(X)$ of S of type k and order $l+1$ has a free set of p elements. For the proof of (ii) it is obviously sufficient to show that the probability of the event A is less than 1 if $p \geqq c_2\sqrt[k]{m \log m}$.

The probability of the event that a given subset of p elements of S is a free set, is

$$\left[\frac{\binom{m-p}{l}}{\binom{m-k}{l}}\right]^{\binom{p}{k}}.$$

Thus the probability of A is less than 1 if

$$\binom{m}{p}\left(\frac{\binom{m-p}{l}}{\binom{m-k}{l}}\right)^{\binom{p}{k}} < 1.$$

It follows that there is a c_2 ($c_2 = c_2(k, l)$) for which the probability of A is less than 1, if $p \geqq c_2 \sqrt[k]{m \log m}$.

Consequently, $p(m, l, k) < c_2 \sqrt[k]{m \log m}$. Q. e. d.

(Received 18 December 1957)

References

[1] P. Erdős, Some remarks on set theory, *Proc. Amer. Math. Soc.*, **1** (1950), pp. 127—141.

[2] P. Erdős and G. Fodor, Some remarks on set theory. VI, *Acta Sci. Math. Szeged*, **18** (1957), pp. 243—260.

[3] A. Tarski, Drei Überdeckungssätze der allgemeinen Mengenlehre, *Fundamenta Math.*, **30** (1938), pp. 132—155.

[4] P. Erdős and A. Tarski, On families of mutually exclusive sets, *Annals of Math.*, **44** (1943), pp. 315—329.

[5] P. Erdős and R. Rado, Combinatorial theorems on classification of subsets of a given set, *Proc. London Math. Soc.* (3), **2** (1952), pp. 417—439.

[6] P. Erdős and R. Rado, A partition calculus in set theory, *Bull. Amer. Math. Soc.*, **62** (1956), pp. 427—489.

[7] K. Kuratowski, Sur une caractérisation des alephs, *Fundamenta Math.*, **38** (1951), pp. 14—17.

[8] G. Fodor, Proof of a conjecture of P. Erdős, *Acta Sci. Math. Szeged*, **14** (1951—1952), pp. 219—227.

[9] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.*, **30** (1930), pp. 264—286.