METRIC PROPERTIES OF POLYNOMIALS

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1. Summary.

Throughout this paper, the symbol f denotes a polynomial

$$(1) \qquad f(z) = \prod_{\nu=1}^{n} (z - z_\nu)$$

(written in the form $f(x) = \Pi(x - x_\nu)$, when only real variables are under consideration). We are concerned primarily with the point set $E = E(f)$, defined as the set where the inequality $|f(z)| < 1$ is satisfied.

In Section 2, we determine the infimum of the length of the longest interval in the set $E \cap L$ (L denotes the real axis) for the case where the x_ν lie on the interval $I = [-1, 1]$ on L. Section 3 deals with the diameter of the set $E \cap L$, under the more general hypothesis that the x_ν lie on the interval $I_r = [-r, r]$.

In Section 4, we study the two-dimensional measure of the set E, under the restriction that all the z_ν lie in the closure $\bar{D}$ of the unit disk D. We use a theorem of G. R. MacLane to show that the measure $|E(f)|$ can be made arbitrarily small. We also deal briefly with the relation between the transfinite diameter of a closed set F and the infimum of $|E(f)|$, under the hypothesis that all z_ν lie in F.

Section 5 is devoted to the problem of finding the greatest number of components that the set E can have when the z_ν are required to lie in $\bar{D}$; Section 6 deals with the sum of the diameters of the components of E, under the hypothesis that all z_ν lie in the disk $D_r : |z| < r$.

Section 7 concerns polynomials (1) for which the set E is connected. In Section 8, we consider two problems concerning the convexity of E and of the components of E, respectively. And in Section 9, we prove a theorem

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on polynomials (1) whose maximum modulus on the unit circle is greater than $(1+c)^n$, where c denotes a positive constant (independent of f).

Many unsolved problems closely related to our theorems are stated in the text.

2. The longest interval of $E \cap L$ (all x_ν on I).

If all the x_ν lie on the interval $I=[-1, 1]$, then the inequality $|E \cap I| \geqq 1$ is satisfied [2, p. 957]; equality holds only in the cases $f(x)=(x \pm 1)^n$. Moreover, Steinberg and others [7] have shown that the set E contains one of the two open halves $(-1, 0)$ and $(0, 1)$ of I. The original proofs of the inequality and of the Steinberg extension were indirect, in the sense that they did not identify any particular segment of the interval I as belonging to the set E. We now give a further extension of the result. Our proof selects one of the open halves of I and shows that it is contained in E.

Theorem 1. *Let the zeros x_ν of the polynomial* (1) *lie in I, and let their centroid $\bar{x}$ lie in $[0, 1]$. Then the set $E \cap L$ contains an interval J which contains the open interval $(0, 1)$; moreover, the interval J contains at least $n/2$ of the x_ν, and $|J| \geqq \sqrt{2}$. On the other hand, the set E does not meet the interval $(-\infty, -\sqrt{2}]$.*

Before giving our proof, we remark that one might reasonably seek an identification, in terms other than the centroid of the x_ν, of an open half of the interval I which lies in E; for E may also contain that half of I which does not contain the centroid of the zeros. In particular, one is tempted to conjecture that E always contains that half of I which contains at least half of the x_ν; but the example $f(x)=(x-1)(x+1/4)^2$ liquidates this proposition.

Our proof of the first part of the theorem depends on the simple fact that the function

$$F(x)=\frac{1}{n}\sum_{\nu=1}^{n}|x-x_\nu|$$

is convex. Since $0 \leqq \bar{x} \leqq 1$, we see at once that $F(1)=1-\bar{x} \leqq 1$; also, since the x_ν lie in I, $F(0) \leqq 1$. Simple considerations show that if

$$F(0) = 1 = F(1),$$

then f has the form $(x^2 - 1)^p$. In this case, the theorem is trivial; we note, incidentally, that here the set $E \cap L$ consists of two open intervals, each of length $\sqrt{2}$. Throughout the remainder of the proof, we assume that f is not of the form $(x^2 - 1)^p$.

The convexity of F implies that $F(x) < 1$ throughout $(0, 1)$. If the x_ν are ordered so that $x_1 \geqq x_2 \geqq \ldots \geqq x_n$, then $F(x)$ assumes its minimum at a point $x = x_h$ whose index satisfies the inequality $h > n/2$, since

$$F'(x) = [M(x) - N(x)]/n,$$

where $M(x)$ and $N(x)$ denote the number of x_ν lying to the left and to the right of x, respectively. The inequality between the geometric and the arithmetic means now gives the result that E contains the intervals $(0, 1)$ and $[x_h, 1)$. The component of $E \cap L$ which contains these two intervals will henceforth be called J.

Next we show that $|J| > \sqrt{2}$. Let J contain the zeros $x_1, x_2, \ldots, x_k$, where $k \geqq h > n/2$, and let x^* be the centroid of these zeros. Let

$$P(x) = \prod_{\nu=k+1}^{n} (x - x_\nu), \quad f_1(x) = (x - x^*)^k P(x).$$

Since $\bar{x}$ is the centroid of the zeros of f_1 as well as of the zeros of f, the interval J_1 corresponding to the polynomial f_1 again contains the interval $(0, 1)$. For any real number x which lies outside the interval J, the relation

$$\frac{1}{k} \sum_{\nu=1}^{k} |x - x_\nu| = |x - x^*|$$

holds, and therefore

$$\prod_{\nu=1}^{k} |x - x_\nu| \leqq |x - x^*|^k,$$

whence $|f(x)| \leqq |f_1(x)|$. It follows that $J \supset J_1$, and hence that $|J| \geqq |J_1|$.

Let $f_2(x) = (x - 1)^k P(x)$. Then the centroid of the zeros of f_2 can not lie to the left of $\bar{x}$, and therefore the interval J_2 corresponding to f_2 contains the interval $(0, 1)$. To see that $|J_1| \geqq |J_2|$, we note that the effect on $|J_1|$ of moving the k zeros at $\bar{x}$ to 1 is the same as would have been the effect of moving all zeros of the factor P to the left through a distance $1-\bar{x}$. Now

$$f_2(\sqrt{2}) = (\sqrt{2}-1)^k P(\sqrt{2}) \leq (\sqrt{2}-1)^k(\sqrt{2}+1)^{n-k} < 1,$$

and it follows that J_2 contains the interval $(0, \sqrt{2}]$.

To prove that the set $E(f)$ does not meet the interval $(-\infty, -\sqrt{2}]$, it is sufficient to prove that $|f(-\sqrt{2})| \geqq 1$. We assert that, for $|x|>1$,

$$(2) \qquad |f(x)| \geq |x-1|^{\lambda}\, |x+1|^{n-\lambda},$$

where $\lambda/n = (1+\bar{x})/2$, that is, where the zeros of the "polynomial" on the right have the same centroid as the zeros of f. To see this, we note that the function $v = \log u$ is concave, and we consider any fixed value x ($|x|>1$). Let $(\bar{u}, \bar{v})$ be the centroid of n unit masses placed at the points $(|x-x_\nu|, \log|x-x_\nu|)$ of the uv-plane, and let $(\bar{U}, \bar{V})$ denote the centroid of two masses of λ and $n-\lambda$ units placed at the points $(|x-1|, \log|x-1|)$ and $(|x+1|, \log|x+1|)$, respectively. Then $\bar{u} = \bar{U}$, by the definition of λ, and therefore $\bar{v} \geqq \bar{V}$; that is,

$$\frac{1}{n}\sum_{\nu=1}^{n} \log|x-x_\nu| \geqq \frac{1}{n}[\lambda \log|x-1| + (n-\lambda)\log|x+1|].$$

The inequality (2) is hereby established. Since $\lambda \geqq n/2$, we conclude that $|f(-\sqrt{2})| \geqq (\sqrt{2}+1)^{\lambda}(\sqrt{2}-1)^{n-\lambda} \geqq 1$, and the proof of Theorem 1 is complete.

For a discussion of the more general problem where the x_ν lie in the interval $I_r = [-r, r]$, see [3].

3. The diameter of $E \cap L$ (all x_ν on I_r).

We now turn our attention to the quantity diam $(E \cap L)$, for the case where the zeros of f lie on the interval I_r. In particular, we consider the supremum $\delta(r)$ of this quantity.

Theorem 2.

$$\delta(r) = 2\sqrt{1+r^2} \qquad (0 \leq r \leq 3/4),$$
$$\delta(r) = 1 + 2r \qquad (3/4 \leq r < \infty).$$

We note that the value $2\sqrt{1+r^2}$ is attained by $|E \cap L|$, in the case $f = x^2 - r^2$; the value $1+2r$ is approached in the case $f = (x-r)^m(x+r)$ (m large). We also note that the theorem remains valid if the zeros of f are restricted merely to the disk $\bar{D}_r$ (see the proof of Theorem 9).

In proving our theorem, it is sufficient to consider the case where all zeros of f lie at the two endpoints of I_r; that is, we need only consider the functions

$$g(x) = |x-r|^m \, |x+r|,$$

where m denotes a positive real number. Corresponding to any other case, we can write an inequality analogous to (2).

To deal with the first part of the theorem, we proceed to show that for $0 \leqq r \leqq 3/4$ the diameter of $E(g) \cap L$ is greatest when $m = 1$.

Let the numbers a and b denote the left and right endpoints of $E(g) \cap L$, and let

$$a = s - \sqrt{1+r^2}, \qquad b = t + \sqrt{1+r^2}.$$

Then, with the notation $R = r + \sqrt{1+r^2}$, we have the relations

$$g(a) = (R-s)^m (R^{-1}-s) = 1,$$
$$g(b) = (R^{-1}+t)^m (R+t) = 1.$$

The quantities $s = s(m)$ and $t = t(m)$ are increasing functions of m, and we wish to show that $s > t$ whenever $m > 1$. We shall do this by defining two functions $m(s)$ and $n(s)$, by means of the equations

$$(R-s)^{m(s)} (R^{-1}-s) = 1,$$
$$(R^{-1}+s)^{n(s)} (R+s) = 1;$$

with this definition, the relations $s(m) > t(m)$ and $n(s) > m(s)$ are equivalent.

The equations defining $m(s)$ and $n(s)$ can be written in the form

$$m(s) = \frac{-\log(R^{-1}-s)}{\log(R-s)} \qquad (0 \leqq s < s_0 = \min[R-1, R^{-1}]),$$

$$n(s) = \frac{\log(R+s)}{-\log(R^{-1}+s)} \qquad (0 \leqq s < s_1 = 1 - R^{-1}).$$

We note that $s_1 \leqq s_0$, since $1 < R \leqq 2$. As s varies from 0 to s_0, the function $m(s)$ increases from 1 to ∞; similarly, as s varies from 0 to s_1, the function $n(s)$ increases from 1 to ∞. Therefore we need only show that $n(s) > m(s)$ whenever $0 < s < s_1$.

We write

$$\begin{aligned}\psi(s) &= -\log(R^{-1}-s)\log(R-s)\,[n(s)-m(s)] \\ &= \log(R+s)\log(R-s) - \log(R^{-1}+s)\log(R^{-1}-s).\end{aligned}$$

For $|u| < 1$,

$$[\log(1+u)\log(1-u)]' = \frac{\log(1-u)}{1+u} - \frac{\log(1+u)}{1-u},$$

and therefore

$$\log(1+u)\log(1-u) = -\sum_{p=1}^{\infty} \frac{a_p u^{2p}}{p},$$

where

$$a_p = 1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{1}{2p-1}.$$

It follows that

$$\psi(s) = \sum_{p=1}^{\infty} \frac{A_p}{p} s^{2p},$$

where

$$A_p = a_p(R^{2p} - R^{-2p}) - (\log R)(R^{2p} + R^{-2p}) \qquad (p = 1, 2, \ldots).$$

The expansion is valid for all values s in the range $0 < s < s_1 = 1 - R^{-1}$, since for such values the inequalities $0 < sR^{-1} < sR < R - 1 \leqq 1$ hold. Clearly, if we can show that all the coefficients A_p are positive, it will follow that $\psi(s) > 0$, and hence that $n(s) > m(s)$.

Now the inequality $A_p > 0$ is equivalent to the condition

$$(3) \qquad k_p(R) = a_p \frac{R^{4p} - 1}{R^{4p} + 1} - \log R > 0.$$

To see that (3) holds, we note first that

$$Rk'_p(R) = \frac{8a_p p R^{4p}}{(R^{4p}+1)^2} - 1;$$

the right member is a decreasing function of R, for $R > 1$, and therefore $k_p(R)$ has no relative minimum in the interval $1 < R < 2$. Since $k_p(1) = 0$, the inequality (3) will be established for the entire range, when we have shown that $k_p(2) > 0$. But

$$a_p - \log 2 > \frac{1}{2p} - \frac{1}{2p+1} = \frac{1}{2p(2p+1)},$$

and therefore

$$k_p(2) = a_p \left\{ \frac{1 - 2^{-4p}}{1 + 2^{-4p}} - 1 \right\} + a_p - \log 2 > -2^{1-4p} + \frac{1}{2p(2p+1)} > 0.$$

This establishes the inequality (3), hence the facts that $\psi(s) > 0$ in the range $0<s<s_1$, that $n(s)>m(s)$ in this range, and that

$$\text{diam}\ [E(g)\cap L] < 2\sqrt{1+r^2}$$

if $m>1$. From this follows the first part of the theorem.

To prove the second part of the theorem we consider, as before, the function $g=|x-r|^m|x+r|$ $(m>0)$. Let s be a positive number such that $g(r+s)=1$. Then $s^m(2r+s)=1$, and therefore $m=-\log(2r+s)/\log s$.

Again, it will be convenient to consider m as a function of s. We shall show that, for $r\geqq 3/4$,

$$g[r+s-(1+2r)]\geqq 1,$$

and from this the desired result will follow immediately.

The last inequality can be written in the form

$$(1+2r-s)^{\frac{\log(2r+s)}{-\log s}}(1-s)\geqq 1.$$

Since $0<s<1$ and $r\geqq 3/4$, the left member of the inequality is a strictly increasing function of r. Since the proof of the first part of the theorem has established the inequality (for each fixed value s in $0<s<1$) in the case $r=3/4$, the inequality holds if $r\geqq 3/4$; we note that equality cannot occur if $r>3/4$. This completes the proof.

The remainder of this section is devoted to a subject which appears to be more difficult: the question of the measure $|E\cap L|$.

Problem 1. To determine the supremum and the infimum of the quantity $|E\cap L|$, under the hypothesis that the x_ν lie on the interval I_r (if no restriction is placed on the x_ν, then the supremum is 4; see [6, p. 229]). The following theorem suggests the conjecture that if all the x_ν lie on I, then $|E\cap L|\leqq 2\sqrt{2}$.

Theorem 3. *If all the zeros of* (1) *lie at the endpoints of I, then $|E\cap L|\leqq 2\sqrt{2}$.*

The case where all the x_ν lie at one endpoint of I is trivial, likewise the case where an equal number of x_ν lie at each endpoint. It is therefore sufficient to consider the function $g(x)=|x+1||x-1|^m$, for $m>1$. Clearly the set $E(g)\cap L$ has exactly two components, and since $g(x)$ takes the value 1 at 0, increases in the interval $[-1,-(m-1)/(m+1)]$, and decreases in the remainder of I, the left-hand component of E has length less than $\sqrt{2}-(m-1)/(m+1)$ (see Theorem 1). Theorem 3 will therefore be proved when we have shown that

$$g\left(\sqrt{2}+\frac{m-1}{m+1}\right)>1 .$$

Now

$$g\left(\sqrt{2}+\frac{m-1}{m+1}\right)=\left(\sqrt{2}+\frac{2m}{m+1}\right)\left(\sqrt{2}-\frac{2}{m+1}\right)^m,$$

and since the first factor is greater than $\sqrt{2}+1$, it suffices to show that the second factor is not less than $\sqrt{2}-1$.

The second factor has the value $\sqrt{2}-1$, when $m=1$. Moreover, its logarithm $h(m)$ has the derivatives

$$h'(m)=\log\left(\sqrt{2}-\frac{2}{m+1}\right)+\frac{m\sqrt{2}}{(m+1)(m+1-\sqrt{2})},$$

$$h''(m)=\frac{2(\sqrt{2}-1)(m-\sqrt{2})}{(m+1)^2(m+1-\sqrt{2})^2}.$$

It follows from the last equation that $h'(m)$ assumes its minimum for $m>1$ at $m=\sqrt{2}$; and since $h'(\sqrt{2})=\log(2-\sqrt{2})+2(\sqrt{2}-1)>0$, it follows that $h(m)$ is an increasing function, for $m>1$; therefore $e^{h(m)}>\sqrt{2}-1$, for $m>1$. This concludes the proof of the theorem.

The infimum of the quantity $|E\cap L|$, for the polynomials f with all x_ν on I, is less than 2; this can be seen from the example $(x+1)(x-1)^m$ $(m\geqq 3)$. Careful computations show that the infimum can not be approached by polynomials of the form $(x-1)^k(x+1)^m$.

We also note that if the zeros of f are merely required to lie on $I_r\,(r\geqq 2)$, then the infimum of $|E(f)\cap L|$ is zero. To see this, it is sufficient to consider the Tchebycheff polynomial $2T_n(x/2)$, and to move each of its zeros which lies outside of $I_{2-\varepsilon}$ to the nearer of the two end-points of I_2. For $r>2$, the minimum value of $|E\cap L|$ for fixed n is $O((2/r)^n)$; for $r=2$, we conjecture that $|E\cap L|>n^{-c}$.

4. The measure of E (all z_ν in $\bar{D}$).

In an earlier paper [2, p. 958], it was stated that if the z_ν lie in the closure of the unit disk D, then the area of the set $E(f)$ exceeds a certain positive universal constant. This statement was based on an erroneous argument (never published) by Mr. Eröd. A recent result of G. R. MacLane [5, Theorem A] shows that the statement is false. For MacLane's theorem implies that if A is any simply connected domain in D whose closure meets neither the origin nor the unit circle C, then there exists an integer $n_0 = n_0(A)$ such that, for each $n > n_0$, some function (1) with all z_ν on C satisfies throughout A the condition $|f(z)| > 2$. Since the area of A can be made arbitrarily close to π, it follows that $\inf |E \cap D| = 0$, for the class of functions (1) with all z_ν on C. We shall now establish an inequaliy which extends this result slightly.

Theorem 4. *If all z_ν lie in $\bar{D}$, then* $|E| \leq 4\{\pi |E \cap D|\}^{1/2}$.

Corollary. *For the class of functions (1) with all z_ν on C,* $\inf |E| = 0$.

Suppose that f has all its zeros in $\bar{D}$. Then the set E does not meet the set $|z| > 2$. Also, it is geometrically clear that

$$|f(re^{i\vartheta})| < |f[(2-r)e^{i\vartheta}]|$$

when $0 < r < 1$. Now let

$$w(z) = w(re^{i\vartheta}) = (2-r)e^{i\vartheta},$$

and let dA_w and dA_z denote differentials of area in the w-plane and in the z-plane, respectively. Then

$$dA_w = \frac{2-r}{r}\, dA_z\,.$$

Since the coefficient of dA_z is a decreasing function of r, it follows that if B denotes a variable region in D, of fixed area S_z, then the area S_w of its image $w(B)$ is at a maximum if B is the disk $|z| < (S_z/\pi)^{1/2}$. In this extreme case,

$$S_w = \pi\left\{4 - \left[2 - \left(\frac{S_z}{\pi}\right)^{1/2}\right]^2\right\} = 4\sqrt{\pi S_z} - S_z\,,$$

and the theorem follows immediately.

Problem 2. To determine the polynomials (1) whose zeros lie in $\bar{D}$ and which have the property that $|E|$ takes on its least value α_n for the fixed degree n. Also, to obtain an estimate of α_n; for example, does there exist a positive constant c such that $\alpha_n > n^{-c}$? We note that the problem of maximizing the quantity $|E|$ has been solved: even without any restriction on the distribution of the z_ν, the inequality $|E| \leqq \pi$ holds, and the supremum is attained only when the z_ν coincide (see Pólya [6, p. 280]). We also call attention to a theorem of H. Cartan [1, p. 273]: the set E can be covered with a family of at most n circular disks the sum of whose radii is less than $2e$.

Problem 3. For the polynomials (1) with all z_ν in $\bar{D}$, let ρ_n denote the radius of the largest disk which is necessarily contained in E. What is the asymptotic behavior of ρ_n? Does there exist a positive constant c such that $\rho_n > c/n$? The example $f(z) = z^n - 1$ shows that the constant c can not be greater than $\pi/2$.

Instead of requiring that the z_ν lie in $\bar{D}$, we may require that they lie on the unit circle C, and we may then ask after the one-dimensional measure of the set $E \cap C$.

Theorem 5. *For a polynomial* (1) *with all z_ν on C, the relation $0 < |E \cap C| < 2\pi$ holds, and the constants 0 and 2π are the best possible.*

We note first that $|E \cap C|$ can not vanish, since (1) has at least one zero on C. Also, $|E \cap C|$ can not have the value 2π, since this would imply that $|f(z)|$ takes its maximum value for $\bar{D}$ at the origin.

To show that $|E \cap C|$ can be made arbitrarily small, we choose a positive number ε, and we consider the auxiliary function $g_n(z) = z^n - 1$. If we modify g_n by moving to $z = 1$ all of its zeros z_ν for which $|\arg z_\nu| < \varepsilon$, we obtain the function

$$f_n(z) = (z^n - 1) \prod{}^{*} \frac{z-1}{z-z_\nu},$$

where the asterisk indicates that the product is to be taken over all indices corresponding to zeros that have been moved. Now, for $0 < \alpha < \vartheta \leqq \pi$,

$$\left| \frac{e^{i\vartheta} - 1}{e^{i\vartheta} - e^{i\alpha}} \right| = \frac{\sin \vartheta/2}{\sin (\vartheta - \alpha)/2},$$

whence

$$\frac{|e^{i\vartheta} - 1|^2}{|e^{i\vartheta} - e^{i\alpha}|\,|e^{i\vartheta} - e^{-i\alpha}|} = \frac{1 - \cos \vartheta}{\cos \alpha - \cos \vartheta} \geqq \frac{2}{1 + \cos \alpha}.$$

This implies that as $n \to \infty$, $|\Pi^*| \to \infty$ uniformly on C, outside of the arc $|\arg z| < \varepsilon$. Therefore $\limsup_{n\to\infty} |E(f_n) \cap C| \leqq 2\varepsilon$, and since ε is arbitrary, our assertion is proved.

That $|E \cap C|$ can be made arbitrarily close to 2π can be proved similarly: the function $g_n(z)$ is modified by moving each of its zeros z_ν with $0 < |\arg z_\nu| < \varepsilon$ to the nearer one of the two points $e^{\pm i\varepsilon}$.

The problems considered in this section can be generalized by replacing the disk D and the circle C by the disk D_r and the circle C_r. An even more general point of view would simply require the zeros to lie on a fixed closed set F. In this connection, we prove the following result.

Theorem 6. *Let F be a closed set of transfinite diameter less than 1. Then there exists a positive number $\rho(F)$ such that, for every polynomial (1) whose zeros lie in F, the set $E(f)$ contains a disk of radius $\rho(F)$.*

Suppose that F is a point set of transfinite diameter less than 1 (see [4, Sections 2 and 3]). Then there exists a finite point set $\{t_j\}_1^m$ $(t_j \in F)$ such that, for $g(z) = \Pi_1^m (z - t_j)$, the inequality $|g(z)| < 1$ holds everywhere in F. Moreover, there exists a number $\rho > 0$ such that $\Pi |z - s_j| < 1$ for all z in F, whenever each point s_j lies in the respective disk H_j of radius ρ and center t_j.

Now let $f(z)$ be any polynomial (1) whose zeros lie in the set F. On the boundary of each of the disks H_j, choose a point s_j at which $|f(z)|$ takes its maximum relative to $\bar{H}_j$. Since

$$\prod_{j=1}^{m} f(s_j) = (-1)^{mn} \prod_{\nu=1}^{n} [(z_\nu - s_1)(z_\nu - s_2) \ldots (z_\nu - s_m)],$$

and since the right member has modulus at most 1, at least one of the quantities $f(s_j)$ has modulus at most 1. It follows that $|f(z)| < 1$ throughout one of the disks H_j, as was to be proved.

Problem 4. Let F be a closed infinite point set, and let $\mu(F)$ denote the infimum of $|E(f)|$, for polynomials (1) with all zeros in F. Is $\mu(F)$ determined by the transfinite diameter of F? In particular, is $\mu(F)$ equal to 0 whenever the transfinite diameter of F is greater than or equal to 1? For the case where F is a line segment or a circular disk, the result follows from the consideration of Tchebycheff polynomials in conjunction with Theorem 8, and from Theorem 4, respectively.

5. The number of components of $\bar{E}$.

From Theorem 1 we see at once that if all the z_ν lie on the interval I, then the number of components of E is at most $1+[n/2]$. If the z_ν are restricted merely to the disk $\bar{D}$, then the number of components of E may be equal to n (example: z^n+1); but the number of components of the closure $\bar{E}$ is at most $n-1$ (see [8]). We shall now show that this result can not be improved, at least when n is sufficiently large.

Theorem 7. *If n is sufficiently large and*

$$P_n(z)=\frac{(z^n+1)(z-1)^2}{(z-e^{i\pi/n})(z-e^{-i\pi/n})},$$

then the set $\bar{E}(P_n)$ has $n-1$ components.

The polynomial P_n is obtained from the polynomial z^n+1 by replacing the two zeros nearest the point $z=1$ by two zeros at $z=1$. To show that it has the required property, we shall divide the plane into $n-1$ domains in such a way that each domain contains one of the zeros of P_n, while $|P_n(z)|>1$ everywhere on the boundaries of the domains except at the origin. Since $P'_n(0)\neq 0$, the origin is not a multiple point of the lemniscate $|P_n(z)|=1$, and it will follow that $\bar{E}(P_n)$ has $n-1$ components.

For each fixed index n, we dissect the plane by means of the arcs

$$\Gamma_n:\ r=\cos\vartheta \qquad \left(2n^{-\frac{1}{3}}\leq|\vartheta|\leq\frac{\pi}{2}\right),$$

$$\Gamma'_n:\ r=1-\frac{|\vartheta|}{2} \qquad \left(\frac{2\pi}{n}\leq|\vartheta|\leq 2n^{-\frac{1}{3}}+\frac{2\pi}{n}\right),$$

and the $n-1$ rays

$$R_{n\nu}:\ z=re^{2\pi i\nu/n} \qquad (\nu=1,2,\ldots,n-1),$$

with the proviso that for those rays $R_{n\nu}$ which lie in the open right-hand plane, the segment between the origin and the first of the curves Γ_n and Γ'_n which is encountered shall be deleted (see Figure 1). Clearly, it will be sufficient to deal with the range $0<\vartheta\leq\pi$.

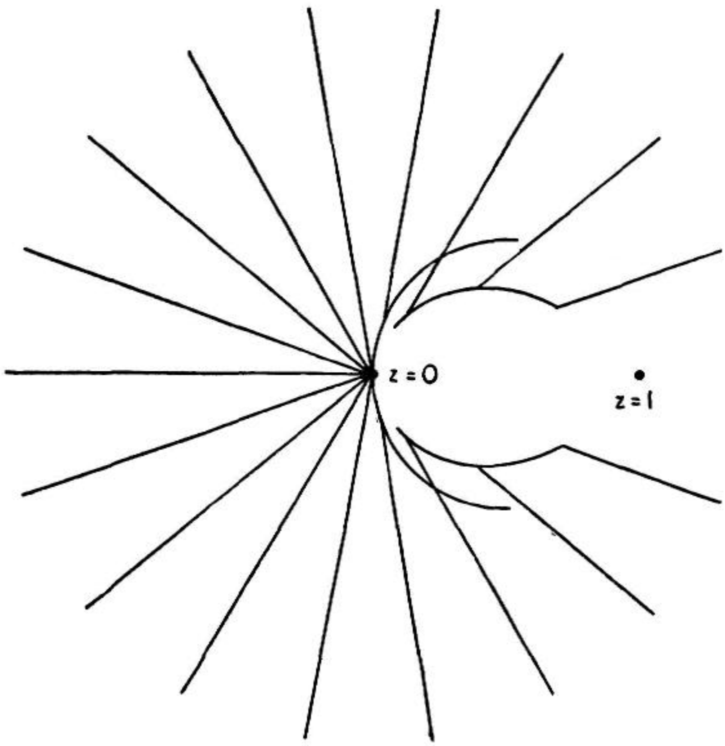

Figure 1. The dissection of the plane for $n = 18$.

For convenience, we write

$$P_n(z) = \frac{z^n + 1}{Q_n(z)}.$$

Then

$$Q_n(z) = \frac{(z - e^{i\pi/n})(z - e^{-i\pi/n})}{(z-1)^2}.$$

To prove our theorem, we shall show that the inequality

$$(4) \qquad |Q_n(z)| < |z^n + 1|$$

holds everywhere on Γ_n, Γ'_n and the segments $R_{n\nu}$, except at the origin.

For $r > 0$, $|Q_n(re^{i\vartheta})|^2 = 1 - 2r\varphi_n(r, \vartheta)$, where

$$(5) \qquad \varphi_n(r, \vartheta) = \frac{2(1 - \cos\pi/n)[r(1 + \cos\pi/n) - (1 + r^2)\cos\vartheta]}{(1 - 2r\cos\vartheta + r^2)^2}.$$

We begin with the rays $R_{n\nu}$ on which $\cos\vartheta \leq 0$, that is, with the rays in the closed left half-plane. Here $\varphi_n(r, \vartheta) > 0$, and therefore

$|Q_n(z)|^2 < 1$. Since $z_n + 1 = r^n + 1$, the inequality (4) holds for the rays $R_{n\nu}$ under consideration.

Next we deal with the curves Γ_n and Γ'_n. For $0 < r < 1$, the inequality $|z^n + 1|^2 > 1 - 2r^n$ is satisfied, and therefore (4) holds provided

$$\varphi_n(r, \vartheta) > r^{n-1}. \tag{6}$$

We shall now show that (6) is satisfied on Γ_n. Since the denominator in the right member of (5) is $\sin^4 \vartheta$, and since the quantity in brackets in the numerator is

$$\cos \vartheta \left(\cos \frac{\pi}{n} - \cos^2 \vartheta \right) = \cos \vartheta \left(\sin^2 \vartheta - 2 \sin^2 \frac{\pi}{2n} \right),$$

(6) will be established when we have shown that

$$4 \sin^2 \frac{\pi}{2n} \left\{ \sin^2 \vartheta - 2 \sin^2 \frac{\pi}{2n} \right\} > \cos^{n-2} \vartheta .$$

Since $2n^{-1/3} \leqq \vartheta \leqq \pi/2$, the left member is greater than $n^{-8/3}$ and the right member less than $\exp(-n^{1/3})$, for all sufficiently large n; therefore (6) is established for all points on Γ_n.

To estimate the value of φ_n on Γ'_n, we note that the denominator in (5) has the value

$$\left[(1-r)^2 + 4r \sin^2 \frac{\vartheta}{2} \right]^2 < \frac{25}{16} \vartheta^4 .$$

The numerator has the value

$$2 \left(1 - \cos \frac{\pi}{n} \right) \left\{ r \left(1 + \cos \frac{\pi}{n} - 2 \cos \vartheta \right) - (1-r)^2 \cos \vartheta \right\}$$

$$= 4 \sin^2 \frac{\pi}{2n} \left\{ \left(1 - \frac{\vartheta}{2} \right) \left(4 \sin^2 \frac{\vartheta}{2} - 2 \sin^2 \frac{\pi}{2n} \right) - \frac{\vartheta^2}{4} \cos \vartheta \right\}$$

$$> (1-\varepsilon) \frac{5 \pi^2 \vartheta^2}{8 n^2} ,$$

for $n > n_\varepsilon$. It follows that $\varphi_n(r, \vartheta) > (1-\varepsilon) 2\pi^2/5n^2\vartheta^2$. On the other hand,

$$r^{n-1} < (1+\varepsilon) \left(1 - \frac{\vartheta}{2} \right)^n < (1+\varepsilon) e^{-n\vartheta/2} .$$

Therefore (6) is satisfied provided $2\pi^2/5n^2\vartheta^2 > e^{-n\vartheta/2}$. With the notation $n\vartheta = 2\pi\lambda$, this relation takes the form $1 > 10\lambda^2 e^{-\pi\lambda}$. Since this inequality holds when $\lambda = 1$, and since the right member decreases monotonically when $\lambda > 2/\pi$, the inequality (6) holds everywhere on Γ'_n, for sufficiently large n.

It remains only to deal with the rays $R_{n\nu}$ in the right half-plane. Let $r_0 e^{i\vartheta}$ be the initial point of such a ray. Then $\varphi_n(r_0, \vartheta) > r_0^{n-1} > 0$, since the point lies on one of the two curves Γ_n and Γ'_n. Also, it is easily seen from (5) that $\varphi_n(1, \vartheta) > 0$. Since the expression in brackets in (5) is a quadratic function of r, and since the coefficient of r^2 is negative, it follows that $\varphi_n(r, \vartheta) > 0$ for $r_0 \leqq r \leqq 1$. Therefore $|Q_n(z)| < 1$, and hence $|P_n(z)| > 1$ on $R_{n\nu} \cap \bar{D}$. Geometrical considerations show that $|P_n(z)|$ increases as z moves from C to ∞ along $R_{n\nu}$. This concludes the proof.

Remarks made at the beginning of this section show that if all zeros of (1) lie in D, then one component of E contains at least two zeros.

Problem 5. If all the z_ν lie in D, does there exist a path of length less than 2 which lies in E and joins two of the z_ν?

Problem 6. Suppose that F is a closed set of transfinite diameter 1, and that F is not contained in any closed disk of radius 1. Can the number of components of E (all $z_\nu \in F$) be n? If the transfinite diameter of F is less than 1, does there exist a positive constant c (depending on F, or perhaps only on the transfinite diameter of F) such that E has at most $(1-c)n$ components, when n is large?

6. The sum of the diameters of the components of E.

The example $f(z) = z^n - 1$ shows that the sum of the diameters of the components of E can be as large as $n2^{1/n}$; we conjecture that it can not be larger, for polynomials (1) of degree n.

If we make the restriction that the zeros of f must lie in the disk $\bar{D}_r$, then, for $n \geqq 3$, the maximum $S(r, n)$ of the sum of the diameters of the components of E appears to be a discontinuous function of r, at $r = 1$. In order to gain some insight into the matter, we study the supremum $S(r)$ of the quantities $S(r, n)$ $(n = 1, 2, \ldots)$. We begin with a preliminary proposition.

Theorem 8. *Let $f(z) = \Pi_1^n (z - x_\nu)$, where the x_ν are real and not all equal; and suppose that a is a positive number such that $|f(a)| = |f(-a)|$. Then $|f(ae^{i\vartheta})| > |f(a)|$, except when $\sin\vartheta = 0$.*

Corollary. *If all the z_ν are real, then the sum of the diameters of the components of E is $|E \cap L|$.*

In proving the theorem, it is clearly sufficient to deal with the range $0<\vartheta<\pi$. Let

$$H(\vartheta) = 2\log|f(ae^{i\vartheta})| = \sum_{\nu=1}^{n} \log(a^2 + x_\nu^2 - 2ax_\nu \cos\vartheta).$$

Then $H'(\vartheta) = 2a\sin\vartheta\, \Sigma x_\nu / |ae^{i\vartheta} - x_\nu|^2$. The quantity $|ae^{i\vartheta} - x_\nu|$ is an increasing or decreasing function of ϑ $(0<\vartheta<\pi)$ according as x_ν is positive or negative. In either case, $x_\nu / |ae^{i\vartheta} - x_\nu|^2$ is a decreasing function of ϑ, and the sum in our expression for $H'(\vartheta)$ is a decreasing function of ϑ. It follows that $H(\vartheta)$ has no relative minimum in the interval $0<\vartheta<\pi$, and the theorem is proved.

Theorem 9. *The function $S(r)$ has the following properties: if $0 \leqq r \leqq 1/2$, then $S(r) = 2\sqrt{1+r^2}$; if $\varepsilon>0$ and b is sufficiently small, then $S(1-b) > (1/2-\varepsilon)(1-e^{-1})\log 1/b$.*

If all the z_ν lie in the disk $\overline{D}_r\,(r \leqq 1/2)$, then the set E contains the disk $D_{1/2}$, and elementary geometrical considerations show further that E is a star-shaped domain. Therefore the sum of the diameters of the components reduces to the diameter of E. Suppose now that for a fixed degree n the z_ν are distributed in such a way as to maximize the diameter of E. Let λ be a line containing a line segment of maximal length with endpoints in $\overline{E}$. Then the z_ν all lie on λ, since otherwise the diameter of E could be increased by replacing each z_ν by its orthogonal projection on λ. The first property now follows from Theorem 2.

Next we consider the auxiliary polynomial $Q(w) = (w^2 - s^2)^q (w+s)$, where q is a positive integer, and where $0<s<1$. For $-s<w<s$,

$$\log|Q(w)| = (2q+1)\log s + q\log\left(1 - \frac{w^2}{s^2}\right) + \log\left(1 + \frac{w}{s}\right);$$

therefore the relation

$$\log\left|Q\left(\frac{s}{2q}\right)\right| = (2q+1)\log s + q\log\left(1 - \frac{1}{4q^2}\right) + \log\left(1 + \frac{1}{2q}\right)$$

holds, and therefore

$$\log\left|Q\left(\frac{s}{2q}\right)\right| > c_1 q \log s + \frac{c_2}{q}$$

when q is sufficiently large [we use the symbols c_i $(i = 1, 2, ...)$ to denote universal positive constants]. Since the right member of the last inequality is 0 when $s = \exp(-c_3/q^2)$ it follows, that with this choice of s the inequality $|Q(w)| > 1$ holds everywhere on the vertical line $\operatorname{Re} w = s/2q$. On the other hand,

$$\log \left| Q\left(\frac{s}{q}\right) \right| = (2q+1)\log s + q \log\left(1 - \frac{1}{q^2}\right) + \log\left(1 + \frac{1}{q}\right)$$

$$< -\frac{1}{q} + \frac{1}{q} = 0 ,$$

and therefore $|Q(w)| < 1$ in the interval $[s/q, s]$. In other words, the set $E(Q)$ has a component which contains the segment $[s/q, s]$ on the real axis and does not meet the line $\operatorname{Re} w = s/2q$.

Now let $f(z) = Q(z^n)$. Then the zeros of f lie on the circle

$$|z| = r = s^{1/n} = \exp \frac{-c_3}{nq^2} .$$

Also, the set $E(f)$ contains n line segments, each in a separate component, and each of length greater than $r(1 - q^{-1/n})$. That is,

$$S\left[\exp \frac{-c_3}{nq^2}\right] > n\left\{\exp \frac{-c_3}{nq^2}\right\}(1 - q^{-1/n}) .$$

If we choose $n = [\log q]$ and write

$$\exp \frac{-c_3}{nq^2} = 1 - b_q = 1 - b ,$$

we obtain the result

$$S(1-b) > [\log q](1-b)\left(1 - \exp \frac{-\log q}{[\log q]}\right) \cdot$$

Now $bq^2[\log q] \to c_3$ as $q \to \infty$, and therefore

$$\log q = \left(\frac{1}{2} + \eta\right)\log \frac{1}{b} ,$$

where $\eta \to 0$ as $q \to \infty$. It follows that if q is sufficiently large, then

$$S(1-b) > \left(\frac{1}{2} - \varepsilon\right)(1 - e^{-1})\log \frac{1}{b} ,$$

for the case where b is one of the numbers b_q. Since $\dfrac{b_{q+1}}{b_q} \to 1$ as $q \to \infty$, the proof is complete.

Problem 7. If all the zeros of f lie in $D_r\,(r<2)$, does the set E have a component with diameter greater than $2-r$?

Problem 8. Let $E(f)$ have the components $E_j\,(j=1, 2, \ldots, k)$ of diameters d_j. Let $\Delta_j = \max\,(0, d_j-1)$, and let $\Delta(f)=\Sigma\Delta_j$. Is $\Delta(f)$ bounded, in the space of polynomials (1)?

Problem 9. Let $N(f)$ denote the number of components of $\bar{E}(f)$ which have diameter greater than 1, and let N_n be the greatest value which $N(f)$ can assume, for polynomials (1) of degree n with all z_ν in D. Is the sequence $\{N_n\}$ bounded?

Problem 10. If f is of the form (1), does there exist a straight line λ such that the projection of $\bar{E}$ on λ has measure at most 2? If z_0 is a point of $\bar{E}$ lying on a line of support of E, does $\bar{E}$ contain a point z such that $|z-z_0|\geqq 2$?

Problem 11. Let $\Lambda(f)$ denote the maximum of the lengths $\lambda_1, \lambda_2, \ldots, \lambda_k$ of the boundaries of the components $E_1, E_2, \ldots, E_k$ of $E(f)$. What is the infimum of $\Lambda(f)$, for all f whose zeros lie in D?

Problem 12. For fixed degree n of f, is the length of the lemniscate $|f(z)|=1$ greatest in the case where $f(z)=z^n-1$? Is the length at least 2π, if E is connected? What is the infimum of the length for polynomials (1) whose zeros all lie in D?

7. Implications of connectedness of E.

A polynomial (1) will be called a K-polynomial provided the set E is connected, a $\bar{K}$-polynomial provided the set $\bar{E}$ is connected. It is easily seen that f is a K-polynomial if and only if $|f(z)|<1$ at all zeros of f', and a $\bar{K}$-polynomial if and only if $|f(z)|\leqq 1$ at all zeros of f'; for if $|f(z)|<A$ at all zeros of f', then none of the lemniscates

$$|f(z)| = B \quad (B\geqq A)$$

has a multiple point, and conversely.

We denote by K_n ($\bar{K}_n$) the class of all K-polynomials ($\bar{K}$-polynomials) of degree n; by K_n^* we denote the class of all polynomials in $\bar{K}_n$ for which $|f(z)|=1$ at all zeros of f'. It is clear that $\bar{K}_n \supset K_n$, $\bar{K}_n \supset K_n^*$, and, for $n \geqq 2$, $K_n \cap K_n^*$ is empty. By $\mathcal{D}(f)$ we denote the discriminant of f. The following theorem establishes a conjecture which was raised by E. Netanyahu and proved, independently, by W. H. Fuchs (oral communication).

Theorem 10. *If $f \in K_n$ $(n > 1)$, then $|\mathcal{D}(f)| < n^n$; if $f \in \overline{K}_n$, then $|\mathcal{D}(f)| \leqq n^n$, and the equality holds if and only if $f \in K_n^*$.*

To prove the theorem, we write f' in the form

$$f'(z) = n \prod_{\nu=1}^{n-1} (z - z'_\nu) .$$

Then

$$|\mathcal{D}(f)| = \prod_{\nu<\mu} |z_\nu - z_\mu|^2 = \prod_{\mu=1}^{n} |f'(z_\mu)| = n^n \prod_{\nu=1}^{n-1} |f(z'_\nu)| .$$

Remark. A similar argument shows that if $\overline{E}$ has n components, then $|\mathcal{D}(f)| > n^n$.

Problem 13. For a fixed value of n, what is the maximum value of $|\mathcal{D}(f)|$ in the space of polynomials (1) with $|z_\mu - z_\nu| \leqq 2$ $(1 \leqq \mu < \nu \leqq n)$? Is the maximum achieved if the z_ν are the vertices of a regular n-gon whose greatest diagonal has length 2?

Problem 14. If f is a K-polynomial, is E contained in a disk of radius 2, and can the center of the disk be placed at the centroid of the zeros?

Problem 15. If f is a $\overline{K}$-polynomial, what are the least possible diameter and the greatest possible width of E? We conjecture that the answer is 2 in both cases (compare Problem 10). What can be said of the sum (or the product) of the diameter and the width?

8. Two problems on convexity.

Theorem 11. *If the z_ν lie in a disk of radius*

$$r_0 = \frac{\sin \pi/8}{1 + \sin \pi/8} ,$$

then the set E is convex.

Suppose that the z_ν lie in the disk $\overline{D}_r$, and let Γ denote the lemniscate $|f(z)| = 1$. We shall show that if $r \leqq r_0$, then Γ can not have a point of inflection.

Suppose that z is a point of inflection of Γ, and that T is the directed tangent to Γ at z. For $\nu = 1, 2, \ldots, n$, let $\rho_\nu = |z - z_\nu|$, and let α_ν denote the angle between the directed line $z_\nu z$ and T. We denote by $z^* = z^*(t)$ the point on Γ which lies at a directed distance t from z, and we proceed to obtain estimates on the distances $|z^* - z_\nu|$.

Since the curve Γ is analytic and has curvature 0 at z, the distance between z^* and the point where the segment $z_\nu z^*$ (possibly extended) meets T is $O(t^3)$. It follows that

$$\begin{aligned} |z_\nu - z^*| &= (\rho_\nu^2 + 2t\rho_\nu \cos\alpha_\nu + t^2)^{1/2} + O(t^3) \\ &= \rho_\nu \left\{1 + \frac{t}{\rho_\nu}\cos\alpha_\nu + \frac{t^2}{2\rho_\nu^2}\sin^2\alpha_\nu + O(t^3)\right\}. \end{aligned}$$

(If the point z_ν happens to lie on T, the point of intersection used in this argument is not defined; but in this case, $\sin\alpha_\nu = 0$, and the estimate on $|z_\nu - z^*|$ is obviously valid.) Since

$$\prod_{\nu=1}^{n} \rho_\nu = 1,$$

it follows further that

$$\begin{aligned} |f(z^*)| &= \prod_{\nu=1}^{n}\left\{1 + \frac{t}{\rho_\nu}\cos\alpha_\nu + \frac{t^2}{2\rho_\nu^2}\sin^2\alpha_\nu + O(t^3)\right\} \\ &= 1 + t\sum_{\nu=1}^{n}\frac{\cos\alpha_\nu}{\rho_\nu} + t^2\left[\sum_{\nu=1}^{n}\frac{\sin^2\alpha_\nu}{2\rho_\nu^2} + \sum_{1\leqq\mu<\nu\leqq n}\frac{\cos\alpha_\mu\cos\alpha_\nu}{\rho_\mu\rho_\nu}\right] + O(t^3). \end{aligned}$$

But since $|f|$ is constant on Γ, the coefficient of t in the last member is 0; and this implies, in turn, that the coefficient of t^2 can be written in the form

$$\sum_{\nu=1}^{n}\frac{\sin^2\alpha_\nu}{2\rho_\nu^2} + \frac{1}{2}\sum_{\mu=1}^{n}\frac{\cos\alpha_\mu}{\rho_\mu}\sum_{\nu=1}^{n}\frac{\cos\alpha_\nu}{\rho_\nu} - \sum_{\nu=1}^{n}\frac{\cos^2\alpha_\nu}{2\rho_\nu^2} = -\sum_{\nu=1}^{n}\frac{\cos 2\alpha_\nu}{2\rho_\nu^2}.$$

Also, it is geometrically obvious that $|z| \geqq 1 - r$, and hence that

$$|\alpha_\mu - \alpha_\nu| \leqq 2\sin^{-1}\frac{r}{1-r},$$

for all μ and ν. Since the coefficient of t is 0, this implies that each of the angles $2\alpha_\nu$ differs from π by less than $4\sin^{-1}\dfrac{r}{1-r}$. We deduce that if

$$\sin^{-1}\frac{r}{1-r} \leqq \frac{\pi}{8},$$

the coefficient of t^2 in the expansion of $|f(z^*)|$ is positive. This contradicts the relation $|f(z^*)| = 1$, for small values of t, and the theorem is proved.

The example $f(z) = (z-r)^m(z+r)$ (m large) shows that Theorem 11 does not remain true if r_0 is replaced by a constant greater than $1/2$.

The following question, raised by H. Grunsky (private communication), is related to our theorem.

Problem 16. Let $\{z_\mu\}_{\mu=1}^m$ be a set of distinct points, and let

$$f(z) = \prod_{\mu=1}^{m}(z-z_\mu)^{k_\mu},$$

where the k_μ are positive integers. Let c be a constant small enough so that the lemniscate $|f(z)| = c$ consists of m distinct loops. Are all the loops convex?

9. Polynomials whose maximum modulus on C is large.

Theorem 12. *For any positive constant c, let $\mathfrak{P}(c)$ denote the class of polynomials (1) whose zeros lie in $\bar{D}$ and whose maximum modulus on C is greater than $(1+c)^n$. Then there exist two constants $c_1 = c_1(c) < 1$ and $c_2 = c_2(c) > 0$ such that, for each f in $\mathfrak{P}(c)$, the inequality $|f(z)| < c_1^n$ holds on a subset of D whose measure is at least c_2.*

Let c be a positive constant for which the theorem is false (without loss of generality, we may assume that $c < 4$). Then there exists a sequence $\{f_j(z)\}$ (f_j of degree n_j) such that $|f_j(1)| > (1+c)^{n_j}$ and such that the measure of the set in D where $|f_j(z)| < c_1^{n_j}$ tends to 0 as $j \to \infty$, for every $c_1 > 0$. We shall show that the existence of such a sequence $\{f_j\}$ entails a contradiction.

It is clear that $n_j \to \infty$. Also, *for $\varepsilon > 0$, the number of zeros of f_j in the disk $|z| \leqq 1-\varepsilon$ can be assumed to be $o(n_j)$*; for otherwise we can select a subsequence of $\{f_j\}$ such that for each f_j in the subsequence the disk $|z| \leqq 1-\varepsilon$ contains at least λn_j of the zeros of f_j; then, throughout the disk $|z| < \eta$, the inequality

$$|f_j(z)| < \{(1-\varepsilon+\eta)^\lambda(1+\eta)^{1-\lambda}\}^{n_j}$$

is satisfied. If $\eta = \eta(\varepsilon, \lambda)$ is sufficiently small, then the content c_1 of the braces is less than 1; that is, $|f_j(z)| < c_1^{n_j}$ throughout the disk D_{η_1}, for each f_j in the subsequence.

For each $\varepsilon > 0$ and each j, let $g_j(z) = g_j(z, \varepsilon) = \Pi(z - z_h)$, where the product ranges over all factors for which z_h is a zero of f_j lying in the domain $|z| > 1 - \varepsilon$. Clearly, the degree m_j of g_j is $n_j - o(n_j)$, and consequently

$$|g_j(1)| \geqq |f_j(1)| \, 2^{-o(n_j)};$$

that is, $|g_j(1)| > (1 + c/2)^{m_j}$, if $n_j > n(\varepsilon)$.

Let $R = R_\varepsilon$ denote the disk $|z - (1 - 5\varepsilon)| < \varepsilon$. We assert that if ε is small enough, then

$$(7) \qquad |g_j(z)| \geqq \left(1 + \frac{c}{4}\right)^{m_j}$$

throughout R. To show this, we consider the ratio

$$\varphi(z, z_h) = \left| \frac{z - z_h}{1 - z_h} \right|,$$

where z lies in R and $1 - \varepsilon < |z_h| \leqq 1$. If z_h lies to the right of the line $x = 1 - 2\varepsilon$, then $\varphi(z, z_h) > 1$. Since the line $x = 1 - 2\varepsilon$ meets the circle $|z| = 1 - \varepsilon$ at $y = \pm\sqrt{\varepsilon(2 - 3\varepsilon)}$, the relations

$$\varphi(z, z_h) = \left| 1 - \frac{1 - z}{1 - z_h} \right| \geqq 1 - \frac{6\varepsilon}{\sqrt{\varepsilon(2 - 3\varepsilon)}}$$

hold for all z_h to the left of the line $x = 1 - 2\varepsilon$. Hence the relation

$$\varphi(z, z_h) > \frac{1 + c/4}{1 + c/2}$$

holds for all z_h and all z in R, provided ε is small enough. In other words, we can choose ε small enough so that (7) holds for all large j and for all z in R_ε.

To obtain our contradiction, it will be sufficient to show that, for $1 - 11\varepsilon/2 < r < 1 - 9\varepsilon/2$, an inequality

$$|f_j(z)| < c_3^{n_j}$$

holds on a set of measure greater than c_4, on the circle $|z| = r$. Since

$$|f_j(z)| \leqq |g_j(z)| \, 2^{n_j - m_j},$$

it is clearly sufficient to prove the corresponding proposition for $g_j(z)$.

Let N_j denote a large integer whose precise value remains yet to be chosen, and let $\zeta_p = ze^{2\pi ip/N_j}$ $(p = 1, 2, ..., N_j)$. Then

$$\prod_{p=1}^{N_j} |g_j(\zeta_p)| = \prod_{p=1}^{N_j} \prod_{h=1}^{m_j} |\zeta_p - z_h|$$

$$= \prod_{h=1}^{m_j} \prod_{p=1}^{N_j} |z_h - \zeta_p| = \prod_{h=1}^{m_j} |z_h^{N_j} - z^{N_j}| \leq (1 + r^{N_j})^{m_j}.$$

Therefore we can choose N_j large enough so that, for all z in our annulus,

$$\prod_{p=1}^{N_j} |g_j(\zeta_p)| < 2. \tag{8}$$

Now the disk R contains at least $c_5 N_j$ of the points ζ_p, where the positive constant c_5 depends on ε, but not on r. For these points, it follows from (7) that

$$|g_j(\zeta_p)| \geq \left(1 + \frac{c}{4}\right)^{m_j};$$

from this it follows that

$$\prod_{p=1}^{N_j} |g_j(\zeta_p)| \geq \left(1 + \frac{c}{4}\right)^{c_5 m_j N_j} \prod{}^{*} |g_j(\zeta_p)|, \tag{9}$$

where the asterisk indicates that all factors for which ζ_p lies in R are to be omitted from the product.

Since the zeros of g_j lie outside of the disk $|z| \leq 1 - \varepsilon$, while $|\zeta_p| = r < 1 - 4\varepsilon$, each of the factors in the left member of (9) is greater than $(3\varepsilon)^{m_j}$. Now let Q_j be the number of factors which are less than $(1 - c/8)^{c_5 m_j}$. Then

$$\prod_{p=1}^{N_j} |g_j(\zeta_p)| \geq \left\{\left(1 + \frac{c}{4}\right)^{c_5} (3\varepsilon)^{Q_j/N_j} \left(1 - \frac{c}{8}\right)^{c_5(1 - Q_j/N_j)}\right\}^{m_j N_j}.$$

Since (8) is satisfied for all z in our annulus, there exists a positive constant μ (independent of z) such that $Q_j/N_j > \mu$ for all j sufficiently large. It follows that on each circle $|z| = r$ in our annulus, the inequality

$$|g_j(z)| < \left(1 - \frac{c}{8}\right)^{c_5 m_j}$$

holds on a set of measure at least $2\pi r\mu$. This completes the proof.

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(*Added in proof*). The sequence $\{N_n\}$ in Problem 9 is not bounded. To see this, it is sufficient to move the two zeros $e^{i\pi/n}$ and $e^{-i\pi/n}$ of the polynomial z^n+1 a short distance δ (along the unit circle) toward the point $z=1$. As this is done, the $[(n-1)/2]$ leaves of the rosette $\overline{E}$ which lie in the left half-plane lose their point of contact (and the other leaves of the rosette coalesce). If $\varepsilon>0$ and δ is small enough, then each of the resulting components of $\overline{E}$ has diameter greater than $2^{1/n}-\varepsilon$, and therefore $N_n\geqq n/2$.

Let the restriction that $|z_\nu|\leqq 1$ be removed, and for $c>0$ let $N_n(c)$ denote the supremum of the number of components of E whose diameter is greater than $1+c$. Is the sequence $\{N_n(c)\}$ bounded, for each fixed value of c?