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ON THE MAXIMUM MODULUS OF ENTIRE FUNCTIONS

By

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The function

$$M(r) = \max_{|z|=r} |f(z)|$$

is called the maximum modulus function of the entire function $f(z)$. In the present paper we discuss the approximation of maximum modulus functions, by means of power series with positive coefficients. We shall prove the following

THEOREM I. *For every $M(r)$ there exists a power series $N(r) = \Sigma c_n r^n$ with non-negative coefficients and with the property*

$$\frac{1}{6} < \frac{M(r)}{N(r)} < 3.$$

Though these constants are not the best possible, the theorem can not be sharpened essentially. We shall show this by constructing a maximum modulus function $M(r)$ with the property that there does not exist a power series $N(r)$ with non-negative coefficients which would satisfy the following asymptotic equality:

$$M(r) \sim N(r).$$

In fact, the following stronger result holds:

THEOREM II. *There exists an absolute constant $\varepsilon_0 > 0$ $\left(\varepsilon_0 = \frac{1}{200}\right)$ and a maximum modulus function $M(r)$ so that for every power series $N(r)$ with non-negative coefficients the inequality*

$$e^{-\varepsilon_0} < \frac{M(r)}{N(r)} < e^{\varepsilon_0}$$

fails for arbitrary large r.

It is to be hoped that by the aid of Theorem I it will be possible to extend certain properties of power series with non-negative coefficients to any maximum modulus function.

Now we turn to the proof of Theorem I. Let

$$f(z)=\sum_{k=0}^{\infty} a_k z^k$$

be an arbitrary entire function, $M(r)$ its maximum modulus function and $F(t)=\log M(e^t)$. Latter is a monotonously increasing, convex, piecewise analytic function in $-\infty<t<+\infty$. In consequence of the convexity, every discontinuity of $F'(t)$ is of the first kind. The following construction of $N(r)$ is based on the approximation by polygons of the curve $F(t)$.

We may suppose without restricting the generality that $a_0 \neq 0$ (if $z=0$ is a λ-fold root of $f(z)$, we can apply the theorem to $f(z)/z^\lambda$). Hence it follows that

$$\lim_{t=-\infty} F'(t)=0, \qquad \lim_{t=-\infty} F(t)=\log|a_0|. \tag{1}$$

Put $t_0=-\infty$, for $n>0$ we define the values t_n so that

$$F'(t_n-0)\leqq n \leqq F'(t_n+0). \tag{2}$$

This defines[1] unambigously the non-decreasing sequence t_n. Now let us define the number τ_n as follows:

Put $\tau_0=t_0=-\infty$ and $n_0=0$. We choose the positive integer k_0 so that

$$F(t_{k_0})-F(\tau_0)\leqq \log 3 < F(t_{k_0+1})-F(\tau_0)$$

and put

$$n_1=\max\{k_0, 1\}, \qquad \tau_1=t_{n_1}.$$

Let us suppose that n_m and $\tau_m=t_{n_m}$ are already defined. Then we put $\tau_{m+1}=t_\nu$ where t_ν has the property that one of the distances $\overline{AB}$ and $\overline{CD}$ (on Fig. 1) is $\leqq \log 3$, but for $t_{\nu-1}$ both are $>\log 3$. However, we have to make an exception if for t_{n_m+1} already both of the distances are $>\log 3$. In this case we put $\tau_{m+1}=t_{n_m+1}$. To formulate the definition, we introduce the following notations:

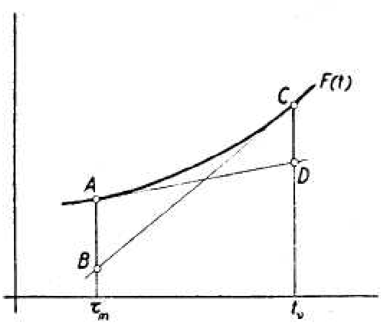

Fig. 1

$$h_\nu^m=\nu(t_\nu-\tau_m)-\{F(t_\nu)-F(\tau_m)\},$$
$$d_\nu^m=\{F(t_\nu)-F(\tau_m)\}-n_m(t_\nu-\tau_m).$$

Owing to the convexity of $F(t)$ these numbers increase with ν and

$$h_{n_m}^m=d_{n_m}^m=0.$$

[1] The numbers t_n are not necessarily different.

We define the integers l_m and k_m in the following way:

$$(3)\qquad \begin{cases} h^m_{l_m} \leqq \log 3 < h^m_{l_m+1}, \\ d^m_{k_m} \leqq \log 3 < d^m_{k_m+1}. \end{cases}$$

After these we define n_{m+1} and τ_{m+1} as follows:

$$(4)\qquad n_{m+1} = \max\{l_m, k_m, n_{m+1}\}, \qquad \tau_{m+1} = t_{n_{m+1}}.$$

The numbers n_m, τ_m and $r_m = e^{\tau_m}$ increase with m monotonously.[2]

We define the positive numbers c_m so that

$$(5)\qquad c_m r_m^{n_m} = M(r_m).$$

Then we shall prove that the power series

$$N(r) = \sum_{m=0}^{\infty} c_m r^m$$

with non-negative coefficients possesses the desired properties.

Before verifying this statement we prove some lemmas in advance.

LEMMA I. *In the interval*

$$t_n \leqq t \leqq t_{n+1}$$

we define the function $G_n(t)$ *as follows:*

$$G_n(t) = \max\{F(t_n) + n(t - t_n);\; F(t_{n+1}) - (n+1)(t_{n+1} - t)\}.$$

Then we have

$$(6)\qquad 0 \leqq F(t) - G_n(t) < \log 3.$$

(Geometrically this states simply that the distance $\overline{PQ}$ on Fig. 2 is $< \log 3$.) This lemma, though simple, is our most difficult one; this is the only place, where we use the fact that $F(t)$ is not the most general monotonic and convex curve, but the logarithm of a maximum modulus function.

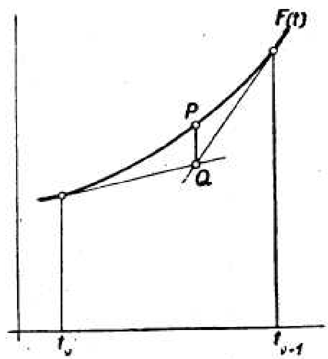

Fig. 2

PROOF OF LEMMA I. The inequality

$$F(t) - G_n(t) \geqq 0$$

follows immediately from the convexity of $F(t)$.

On the other hand, let us suppose that the second half of (6) is false, i. e. in the interval

$$t_n \leqq t \leqq t_{n+1}$$

there exists a point $\bar{t}$ for which

$$(6^*)\qquad F(\bar{t}) - G_n(\bar{t}) \geqq \log 3.$$

[2] It is possible that — at most — two τ_m coincide.

From our hypothesis it follows that

$$(7)\qquad \log 3 \leqq F(\bar t)-G_n(\bar t) \leqq \int_{t_n}^{\bar t}(F'(t_n)-n)\,dt \leqq \int_{t_n}^{\bar t} dt=\bar t-t_n,$$

$$(8)\qquad \log 3 \leqq F(\bar t)-G_n(\bar t) \leqq \int_{\bar t}^{t_{n+1}}(n+1-F'(t))\,dt \leqq \int_{\bar t}^{t_{n+1}} dt=t_{n+1}-\bar t.$$

We introduce the following notations:

$$\varrho_n=e^{t_n}, \qquad \bar\varrho=e^{\bar t}.$$

Using Cauchy's inequality we have

$$|a_k|\varrho_n^k \leqq M(\varrho_n), \qquad |a_k|\varrho_{n+1}^k \leqq M(\varrho_{n+1})$$

or

$$(9)\qquad \log|a_k|+kt_n \leqq F(t_n), \qquad \log|a_k|+kt_{n+1} \leqq F(t_{n+1}).$$

Hence, by (6*), (7), (9), we obtain

$$\log|a_k\bar\varrho^k|=\log|a_k|+k\bar t=\log|a_k|+kt_n+k(\bar t-t_n) \leqq F(t_n)+k(\bar t-t_n) \leqq$$
$$\leqq G_n(\bar t)-(n-k)(\bar t-t_n) \leqq F(\bar t)-\log 3-\log 3\,(n-k)=$$
$$=F(\bar t)-\log 3\,(n+1-k)=\log M(\bar\varrho)-\log 3\,(n+1-k) \quad \text{for} \quad k\leqq n,$$

i. e.:

$$(10)\qquad \frac{|a_k|\bar\varrho^k}{M(\bar\varrho)} \leqq 3^{-(n+1-k)} \quad \text{for} \quad k\leqq n.$$

Similarly, using (8), we obtain

$$\log|a_k\bar\varrho^k|=\log|a_k|+k\bar t=\log|a_k|+kt_{n+1}-k(t_{n+1}-\bar t) \leqq$$
$$\leqq F(t_{n+1})-k(t_{n+1}-\bar t) \leqq G_n(\bar t)-(k-n-1)(t_{n+1}-\bar t) \leqq$$
$$\leqq F(\bar t)-\log 3-\log 3\,(k-n-1)=\log M(\bar\varrho)-\log 3\,(k-n) \quad \text{for} \quad k\geqq n+1,$$

i. e.:

$$(11)\qquad \frac{|a_k|\bar\varrho^k}{M(\bar\varrho)} \leqq 3^{-(k-n)} \quad \text{for} \quad k\geqq n+1.$$

By virtue of (10) and (11)

$$\frac{\sum_{k=0}^{\infty}|a_k|\bar\varrho^k}{M(\bar\varrho)} < 2\left(\frac{1}{3}+\frac{1}{3^2}+\cdots\right)=1$$

which is impossible.

LEMMA II. a) *Let us consider the lines*

$$u = n_k t + \log c_k \qquad (k = 0, 1, 2, \ldots) \tag{12}$$

and their upper supporting curve $G(t)$. If we denote the maximal term of $N(r)$ with $\mu(r)$, then

$$G(t) = \log \mu(t).$$

b) *Since in view of* (5)

$$\log c_k + n_k \tau_k = \log M(r_k) = F(\tau_k), \tag{13}$$

therefore (12) *is the supporting line of the curve $F(t)$ at the point $t = t_k$. So $G(t)$ is a convex poligon which touches (or rather supports) the curve $F(t)$ at the points $t = \tau_k$ $(k = 0, 1, 2, \ldots)$.*

LEMMA III.

$$0 \leqq F(t) - G(t) \leqq \log 3.$$

PROOF. Let be $\tau_m \leqq t \leqq \tau_{m+1}$. Suppose first that $n_{m+1} = l_m$. Then, by virtue of (13) and Lemma II, using the convexity of $F(t)$, we can write

$$\begin{aligned} 0 &\leqq F(t) - G(t) \leqq F(t) - \{n_{m+1} t + \log c_{m+1}\} = \\ &= F(t) - \{n_{m+1}\tau_{m+1} + \log c_{m+1}\} + n_{m+1}(\tau_{m+1} - t) = \\ &= F(t) - F(\tau_{m+1}) + n_{m+1}(\tau_{m+1} - t) \leqq n_{m+1}(\tau_{m+1} - \tau_m) - \\ &- \{F(\tau_{m+1}) - F(\tau_m)\} = h^m_{n_{m+1}} = h^m_{l_m} \leqq \log 3. \end{aligned}$$

In the same way we obtain

$$0 \leqq F(t) - G(t) \leqq d^m_{n_{m+1}} = d^m_{k_m} \leqq \log 3$$

also in the case $n_{m+1} = k_m$. Finally, in the case $n_{m+1} = n_m + 1$ the statement of the lemma follows immediately from Lemma I, because in that case

$$G(t) \equiv G_{n_m}(t)$$

for $\tau_m = t_{n_m} \leqq t \leqq t_{n_{m+1}} = \tau_{m+1}$.

LEMMA IV. *We introduce for $m < p$ the following notations:*

$$\begin{aligned} D_{m,p} &= \{F(\tau_p) - F(\tau_m)\} - n_m(\tau_p - \tau_m), \\ H_{m,p} &= n_p(\tau_p - \tau_m) - \{F(\tau_p) - F(\tau_m)\}. \end{aligned}$$

(We mention that $D_{m,m+1} = d^m_{n_m+1}$, $H_{m,m+1} = h^m_{n_m+1}$.) Then, in consequence of the convexity of $F(t)$, for $m < p < s$ we obtain

$$\begin{aligned} D_{m,s} &= \{F(\tau_s) - F(\tau_m)\} - n_m(\tau_s - \tau_m) \geqq \\ &\geqq \{F(\tau_s) - F(\tau_p)\} + \{F(\tau_p) - F(\tau_m)\} - n_m(\tau_p - \tau_m) - n_p(\tau_s - \tau_p) = D_{m,p} + D_{p,s} \end{aligned}$$

and similarly

$$H_{m,s} \geqq H_{m,p} + H_{p,s}.$$

LEMMA V.

$$D_{m,m+2} \geqq \log 3, \qquad H_{m,m+2} \geqq \log 3.$$

Namely, in view of (3) and (4), in consequence of the convexity of $F(t)$,

$$D_{m,m+2} = \{F(\tau_{m+2}) - F(\tau_m)\} - n_m(\tau_{m+2} - \tau_m) \geqq$$
$$\geqq \{F(t_{n_{m+1}+1}) - F(\tau_m)\} - n_m(t_{n_{m+1}+1} - \tau_m) \geqq$$
$$\geqq \{F(t_{k_m+1}) - F(\tau_m)\} - n_m(t_{k_m+1} - \tau_m) = d^{m}_{k_m+1} > \log 3$$

and the assertion on $H_{m,m+2}$ follows in the same way.

LEMMA VI. *For $k > 0$, on the basis of Lemma IV and V we have*

$$D_{m-(2k+1),m} \geqq D_{m-2k,m} \geqq \sum_{i=0}^{k-1} D_{m-2(i+1),m-2i} \geqq k \log 3$$

and in the same way

$$H_{m,m+2k+1} \geqq H_{m,m+2k} \geqq \sum_{i=1}^{k} H_{m+(i-1)2,m+2i} \geqq k \log 3.$$

LEMMA VII. *For $k > 0$ we have*

$$\frac{c_{m-2k-1} r_m^{n_{m-2k-1}}}{c_m r_m^{n_m}} \leqq \frac{c_{m-2k} r_m^{n_{m-2k}}}{c_m r_m^{n_m}} \leqq 3^{-k},$$

$$\frac{c_{m+2k+1} r_m^{n_{m+2k+1}}}{c_m r_m^{n_m}} \leqq \frac{c_{m+2k} r_m^{n_{m+2k}}}{c_m r_m^{n_m}} \leqq 3^{-k}.$$

The lemma follows immediately from the previous one, by considering that in view of (13)

$$\log \frac{c_\nu r_m^{n_\nu}}{c_m r_m^{n_m}} = \log c_\nu - \log c_m + n_\nu \tau_m - n_m \tau_m =$$

$$= n_\nu(\tau_m - \tau_\nu) - \{F(\tau_m) - F(\tau_\nu)\} = \begin{cases} -D_{\nu,m} & \text{if} \quad \nu < m, \\ -H_{m,\nu} & \text{if} \quad \nu > m. \end{cases}$$

LEMMA VIII. *For $r_m \leqq r \leqq r_{m+1}$ we have*

$$0 < N(r) - \{c_{m-1} r^{n_{m-1}} + c_m r^{n_m} + c_{m+1} r^{n_{m+1}} + c_{m+2} r^{n_{m+2}}\} \leqq 2\mu(r).$$

Namely,[3] in view of the previous lemma

$$0 < N(r) - \{c_{m-1} r^{n_{m-1}} + c_m r^{n_m} + c_{m+1} r^{n_{m+1}} + c_{m+2} r^{n_{m+2}}\} =$$

$$= \sum_{\nu=0}^{m-2} c_\nu r^{n_\nu} + \sum_{\nu=m+3}^{\infty} c_\nu r^{n_\nu} = c_m r^{n_m} \sum_{\nu=1}^{m-2} \frac{c_\nu r^{n_\nu}}{c_m r^{n_m}} + c_{m+1} r^{n_{m+1}} \sum_{\nu=m+3}^{\infty} \frac{c_\nu r^{n_\nu}}{c_{m+1} r^{n_{m+1}}} \leqq$$

$$\leqq c_m r^{n_m} \sum_{\nu=1}^{m-2} \frac{c_\nu r_m^{n_\nu}}{c_m r_m^{n_m}} + c_{m+1} r^{n_{m+1}} \sum_{\nu=m+2}^{\infty} \frac{c_\nu r_{m+1}^{n_\nu}}{c_{m+1} r_{m+1}^{n_{m+1}}} \leqq$$

$$\leqq c_m r^{n_m} \sum_{k=1}^{[m/2]} 2 \cdot 3^{-k} + c_{m+1} r^{n_{m+1}} \sum_{k=1}^{\infty} 2 \cdot 3^{-k} \leqq$$

$$\leqq (c_m r^{n_m} + c_{m+1} r^{n_{m+1}})\, 2 \left(\frac{1}{3} + \frac{1}{3^2} + \cdots\right) \leqq 2\mu(r).$$

LEMMA IX.

$$\mu(r) < N(r) < 6\mu(r).$$

Namely, by virtue of the previous lemma we have

$$N(r) = [N(r) - \{c_{m-1} r^{n_{m-1}} + c_m r^{n_m} + c_{m+1} r^{n_{m+1}} + c_{m+2} r^{n_{m+2}}\}] +$$
$$+ \{c_{m-1} r^{n_{m-1}} + c_m r^{n_m} + c_{m+1} r^{n_{m+1}} + c_{m+2} r^{n_{m+2}}\} \leqq 2\mu(r) + 4\mu(r) = 6\mu(r).$$

After these preliminary remarks Theorem I follows immediately. In fact, from Lemma IX we get

$$(14) \qquad \frac{1}{6} < \frac{\mu(r)}{N(r)} < 1.$$

On the other hand, in view of Lemma II and III ($t = \log r$) we have

$$(15) \qquad 1 \leqq \frac{M(r)}{\mu(r)} \leqq 3.$$

By comparing (14) and (15) we obtain the desired inequality:

$$\frac{1}{6} < \frac{M(r)}{N(r)} < 3.$$

Q. e. d.

Now we turn to Theorem 2. We define the entire function $f(z)$ by the power series

$$(16) \qquad f(z) = \sum_{k=0}^{\infty} \frac{z^{n_k}}{2^{n_k^2}} \left(1 + \frac{z}{r_k} - \frac{z^2}{r_k^2}\right)$$

where $n_k = 2^k$, $r_k = 4^{n_k}$. Let $M(r)$ be the maximum modulus function of $f(z)$.

[3] The following calculation is, strictly speaking, restricted to the case $m \geqq 2$. However, if we put $c_{-1} = c_{-2} = 0$, then we can apply the argument to the case $m = 0$, $m = 1$, too.

We shall demonstrate that there does not exist a power series

$$N(r)=\sum_{n=0}^{\infty} a_n r^n$$

with non-negative coefficients which satisfies the inequality

$$e^{-\varepsilon}<\frac{N(r)}{M(r)}<e^{\varepsilon} \tag{17}$$

with $\varepsilon \leqq \frac{1}{200}$ and for $r>r_{k_0-1}$. Let us suppose that our assertion is false and there exists an $N(r)$ satisfying (17). We introduce the following notations:

$$\left.\begin{aligned} \varrho_k &= r_k^{5/6} r_{k+1}^{1/6}, & \tau_k &= r_k^{2/3} r_{k+1}^{1/3}, \\ \sigma_k &= r_k^{1/3} r_{k+1}^{2/3}, & \zeta_k &= r_k^{1/6} r_{k+1}^{5/6} \end{aligned}\right\} \qquad (k=0,1,2,\ldots).$$

In consequence of the definition we obtain

$$r_k<\varrho_k<\tau_k<\sigma_k<\zeta_k<r_{k+1}.$$

Let us put $\sigma_{k-1} \leqq r \leqq \tau_k$. Then, for $\nu>0$,

$$\log \frac{\dfrac{r^{n_{k-\nu}+2}}{2^{n_{k-\nu}^2}}}{\dfrac{r^{n_k}}{2^{n_k^2}}}=(n_k^2-n_{k-\nu}^2)\log 2-(n_k-n_{k-\nu}-2)\log r \leqq$$

$$\leqq (n_k-n_{k-\nu})\left\{\log 2(n_k+n_{k-\nu})-\log \sigma_{k-1}\left(1-\frac{2}{n_k-n_{k-\nu}}\right)\right\}= \tag{18}$$

$$=(n_k-n_{k-\nu})\left\{\log 2\,(n_k+n_{k-\nu})-\log 4\,\frac{n_{k-1}+n_k}{3}\right\}+2\log 4\,\frac{n_{k-1}+2n_k}{3} \leqq$$

$$\leqq (n_k-n_{k-\nu})\left\{\log 2(n_k+n_{k-\nu})-\frac{2}{3}\log 2(n_{k-\nu}+2n_k)\right\}+4\log 2\cdot n_k=$$

$$=-(n_k-n_{k-\nu})^2\frac{\log 2}{3}+4\log 2\cdot n_k \leqq -(n_k-n_{k-1})^2\frac{\log 2}{3}+4\log 2\cdot n_k=$$

$$=-4^{k-1}\frac{\log 2}{3}+\log 2\cdot 2^{k+2}<-4^{k-3}.$$

We obtain in the same way that in the same interval

$$\log \frac{\dfrac{r^{n_{k+\nu}}}{2^{n_{k+\nu}^2}}}{\dfrac{r^{n_k}}{2^{n_k^2}}}<-(n_{k+\nu}-n_k)^2\frac{\log 2}{3}<-4^{k+\nu-1}\frac{\log 2}{3}<-4^{k-3+\nu}. \tag{19}$$

From (18) and (19) it follows that in the interval $\sigma_{k-1} \leqq r \leqq \tau_k$ the k-th term of (16) (for sufficiently large k) predominates strongly over the rest. Therefore, denoting the maximum modulus function of the polynomial

$$1+\zeta-\zeta^2$$

by $M_1(r)$, we have

$$e^{-\varepsilon} < \frac{\dfrac{r^{n_k}}{2^{n_k^2}} M_1\left(\dfrac{r}{r_k}\right)}{M(r)} < e^{\varepsilon} \tag{20}$$

in the interval $\sigma_{k-1} \leqq r \leqq \tau_k$ for $k > k_1(\varepsilon)$. Comparing (17) and (20) we obtain that in the same interval

$$e^{-2\varepsilon} < \frac{r^{n_k}}{2^{n_k^2}} \frac{M_1\left(\dfrac{r}{r_k}\right)}{N(r)} < e^{2\varepsilon} \tag{21}$$

holds for $k > k_2(\varepsilon) = \max\{k_0(\varepsilon), k_1(\varepsilon)\}$. On the other hand, in the interval $\sigma_{k-1} \leqq r \leqq \zeta_{k-1}$

$$\begin{aligned} 1 < M_1\left(\frac{r}{r_k}\right) < M_1\left(\frac{\zeta_{k-1}}{r_k}\right) = M_1\left(\left\{\frac{r_{k-1}}{r_k}\right\}^{\frac{1}{6}}\right) = M_1\left(4^{-\frac{n_k - n_{k-1}}{6}}\right) = \\ = M_1\left(4^{-\frac{2^{k-2}}{3}}\right) < e^{\varepsilon} \quad \text{if} \quad k > k_3(\varepsilon). \end{aligned} \tag{22}$$

In the same way in the interval $\varrho_k \leqq r \leqq \tau_k$

$$1 < \frac{r_k^2}{r^2} M_1\left(\frac{r}{r_k}\right) < e^{\varepsilon} \quad \text{if} \quad k > k_3(\varepsilon). \tag{23}$$

Comparing (21) with (22) and (23), respectively, we obtain the following inequalities:

$$e^{-3\varepsilon} < \frac{r^{n_k}}{2^{n_k^2}} \frac{1}{N(r)} < e^{2\varepsilon}, \tag{24}$$

$$e^{-3\varepsilon} < \frac{r^{n_k+2}}{2^{n_k^2} r_k^2} \frac{1}{N(r)} < e^{2\varepsilon} \tag{25}$$

which are valid in the interval

$$\sigma_{k-1} \leqq r \leqq \zeta_{k-1} \quad \text{and} \quad \varrho_k \leqq r \leqq \tau_k,$$

respectively. Applying (24) for $r=\xi_{k-1}=\sqrt{\sigma_{k-1}\zeta_{k-1}}$, we obtain for $n<n_k$

$$a_n\xi_{k-1}^n=(a_n\sigma_{k-1}^n)\left(\frac{\xi_{k-1}}{\sigma_{k-1}}\right)^n\leqq N(\sigma_{k-1})\left(\frac{\xi_{k-1}}{\sigma_{k-1}}\right)^n<$$

$$<e^{3\varepsilon}\frac{\sigma_{k-1}^{n_k}}{2^{n_k^2}}\left(\frac{\xi_{k-1}}{\sigma_{k-1}}\right)^n=e^{3\varepsilon}\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}\left(\frac{\sigma_{k-1}}{\xi_{k-1}}\right)^{n_k-n}\leqq \tag{26}$$

$$\leqq e^{3\varepsilon}\left(\frac{r_{k-1}}{r_k}\right)^{\frac{n_k-n}{12}}\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}<\frac{2}{2^{\frac{n_k-n_{k-1}}{6}}}\cdot\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}=2^{1-\frac{n_{k-1}}{6}}\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}$$

and similarly for $n>n_k$

$$a_n\xi_{k-1}^n=(a_n\zeta_{k-1}^n)\left(\frac{\xi_{k-1}}{\zeta_{k-1}}\right)^n\leqq N(\zeta_{k-1})\left(\frac{\xi_{k-1}}{\zeta_{k-1}}\right)^n<$$

$$<e^{3\varepsilon}\frac{\zeta_{k-1}^{n_k}}{2^{n_k^2}}\left(\frac{\xi_{k-1}}{\zeta_{k-1}}\right)^n=e^{3\varepsilon}\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}\left(\frac{\xi_{k-1}}{\zeta_{k-1}}\right)^{n-n_k}\leqq \tag{27}$$

$$\leqq e^{3\varepsilon}\left(\frac{r_{k-1}}{r_k}\right)^{\frac{n-n_k}{12}}\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}\leqq 2^{1-(n-n_k)\frac{n_{k-1}}{6}}\cdot\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}.$$

Here we utilized that the coefficients a_n are non-negative. From (26) and (27) we have

$$0<\sum_{n\neq n_k}a_n\xi_{k-1}^{n_k}=N(\xi_{k-1})-a_{n_k}\xi_{k-1}^{n_k}\leqq$$

$$\leqq\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}\left\{n_k2^{1-\frac{n_{k-1}}{6}}+\frac{2}{2^{\frac{n_{k-1}}{6}}-1}\right\}<\varepsilon\frac{\xi_{k-1}^{n_k}}{2^{n_k^2}}, \tag{28}$$

$$0<\frac{N(\xi_{k-1})}{\xi_{k-1}^{n_k}}2^{n_k^2}-a_{n_k}2^{n_k^2}<\varepsilon\quad\text{if}\quad k>k_4(\varepsilon).$$

On the other hand, it follows from (24) that

$$e^{-2\varepsilon}<\frac{N(\xi_{k-1})}{\xi_{k-1}^{n_k}}2^{n_k^2}<e^{3\varepsilon}. \tag{29}$$

Comparing (28) and (29) we obtain

$$e^{-2\varepsilon}-\varepsilon<a_{n_k}2^{n_k^2}<e^{3\varepsilon}. \tag{30}$$

Using (25) we obtain in an entirely similar way that

$$e^{-2\varepsilon}-\varepsilon<a_{n_k+2}2^{n_k^2}<e^{3\varepsilon}. \tag{31}$$

From (26) it follows immediately that

$$a_n r^n < 2^{1-\frac{n_k-1}{6}} \frac{r^{n_k}}{2^{n_k^2}} \tag{32}$$

for $r \geqq \zeta_{k-1} > \xi_{k-1}$, $n < n_k$ and in an entirely similar way we also obtain

$$a_n r^n < 2^{1-(n-n_k)\frac{n_k}{6}} \frac{r^{n_k+2}}{2^{n_k^2} r_k^2} \tag{33}$$

for $r \leqq \varrho_k$, $n > n_k + 2$. From (32) and (33) it follows that in the interval $\zeta_{k-1} \leqq r \leqq \varrho_k$

$$\begin{aligned} 0 < N(r) - \{a_{n_k} r^{n_k} + a_{n_k+1} r^{n_k+1} + a_{n_k+2} r^{n_k+2}\} = \\ = \sum_{\nu=0}^{n_k-1} a_\nu r^\nu + \sum_{\nu=n_k+3}^{\infty} a_\nu r^\nu \leqq \left\{ n_k \cdot 2^{1-\frac{n_k-1}{6}} + \frac{2}{2^{\frac{n_k-1}{6}}-1} \frac{r^2}{r_k^2} \right\} \frac{r^{n_k}}{2^{n_k^2}} \leqq \\ \leqq \varepsilon \left(1 + \frac{r^2}{r_k^2}\right) \frac{r^{n_k}}{2^{n_k^2}} \quad \text{if} \quad k > k_4(\varepsilon). \end{aligned} \tag{34}$$

From (30) and (31) we obtain

$$\left(1+\frac{r^2}{r_k^2}\right)(e^{-2\varepsilon}-\varepsilon) < (a_{n_k} + a_{n_k+2} r^2) 2^{n_k^2} < \left(1+\frac{r^2}{r_k^2}\right) e^{3\varepsilon}.$$

Hence

$$\begin{aligned} r^{n_k}(e^{-2\varepsilon}-\varepsilon) \left\{ \frac{1}{2^{n_k^2}} \left(1+\frac{r^2}{r_k^2}\right) + a_{n_k+1} r \right\} < \left\{ a_{n_k} r^{n_k} + a_{n_k+1} r^{n_k+1} + a_{n_k+2} r^{n_k+2} \right\} < \\ < e^{3\varepsilon} \left\{ \frac{1}{2^{n_k^2}} \left(1+\frac{r^2}{r_k^2}\right) + a_{n_k+1} r \right\} r^{n_k}. \end{aligned} \tag{35}$$

In view of (34) and (35)

$$e^{-2\varepsilon} - \varepsilon < \frac{N(r)}{\left\{ \frac{1}{2^{n_k^2}} \left(1+\frac{r^2}{r_k^2}\right) + a_{n_k+1} r \right\} r^{n_k}} < e^{3\varepsilon} + \varepsilon. \tag{36}$$

Comparing this with (21) we obtain

$$\begin{aligned} e^{-2\varepsilon}(e^{-2\varepsilon}-\varepsilon) < \frac{M_1\left(\frac{r}{r_k}\right)}{1 + 2^{n_k^2} a_{n_k+1} r + \frac{r^2}{r_k^2}} < e^{2\varepsilon}(e^{3\varepsilon}+\varepsilon), \\ \left| \frac{1 + 2^{n_k^2} a_{n_k+1} r + \frac{r^2}{r_k^2}}{M_1\left(\frac{r}{r_k}\right)} - 1 \right| < 7\varepsilon \quad \text{if} \quad \varepsilon < \frac{1}{20}. \end{aligned} \tag{37}$$

It is easy to verify that

$$M_1(r)=\begin{cases} 1+r-r^2 & \text{if} \quad 0\leqq r\leqq \sqrt{5}-2, \\ \frac{\sqrt{5}}{2}(1+r^2) & \text{if} \quad \sqrt{5}-2\leqq r\leqq \sqrt{5}+2, \\ r^2+r-1 & \text{if} \quad \sqrt{5}+2\leqq r<\infty. \end{cases}$$

Thus substituting in (37) $r=r_k$ and $r=\frac{r_k}{10}$, respectively, we obtain ($M_1(1)=\sqrt{5}$, $M_1(0{,}1)=1{,}09$)

$$\left|\frac{2+2^{n_k^2}a_{n_k+1}r_k}{\sqrt{5}}-1\right|<7\varepsilon, \tag{38}$$

$$\left|\frac{1{,}01+0{,}1\cdot 2^{n_k^2}a_{n_k+1}r_k}{1{,}09}-1\right|<7\varepsilon. \tag{39}$$

From (38) we have

$$2^{n_k^2}a_{n_k+1}r_k<(\sqrt{5}-2)+7\sqrt{5}\,\varepsilon,$$

on the other hand, from (39) we obtain

$$2^{n_k^2}a_{n_k+1}r_k>0{,}8-76{,}3\varepsilon.$$

If $\varepsilon\leqq\frac{1}{200}$, these inequalities are inconsistent.

Thus, we arrived at a contradiction and this proves Theorem II.

It is an open question whether to an arbitrary maximum modulus function $M(r)$ there exists a power series $V(r)=\Sigma a_n r^n$ with real coefficients, and with the maximum modulus function $M^*(r)$ with the property that either

$$M(r)\sim V(r)$$

or

$$M(r)\sim M^*(r)$$

holds.

Similarly, the authors do not know whether Theorem I holds for every piecewise smooth, non-decreasing, logarithmically convex function $\mathfrak{M}(r)$, or not.

(Received 27 August 1956)

О МАКСИМУМЕ МОДУЛЯ ЦЕЛЫХ ФУНКЦИЙ

П. Эрдёш и Т. Кёвари (Будапешт)

(Резюме)

В настоящей статье доказываются следующие две теоремы:

Теорема I. Если $f(z)$ любая целая функция, $M(r)=\max\limits_{|z|=r}|f(z)|$, то существует такой степенной ряд с неотрицательными коэффициентами $N(r)=\sum c_n r^n$, что

$$\frac{1}{6}<\frac{M(r)}{N(r)}<3 \qquad (0\leq r<\infty).$$

Эти постоянные не наилучшие, но теорема в сущности все же не может быть усилена, ибо имеет место следующая теорема:

Теорема II. Предыдущая теорема не будет иметь место даже для достаточно больших r, если заменить в ней $\frac{1}{6}$ на $e^{-\frac{1}{200}}$ и 3 на $e^{\frac{1}{200}}$.

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