ON A HIGH-INDICES THEOREM IN BOREL SUMMABILITY

By

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To the memory of O. SZÁSZ

Let $\sum_{k=0}^{\infty} a_k$ be an infinite series. Put $a'_k = \frac{1}{2^{k+1}} \sum_{i=0}^{k} \binom{k}{i} a_k$. If $\sum_{k=0}^{\infty} a'_k$ converges, it is defined as the Euler sum of $\sum_{k=0}^{\infty} a_k$. It is easy to see that if $\sum_{k=0}^{\infty} a_k$ converges, so does $\sum_{k=0}^{\infty} a'_k$, i. e. Euler summability is regular. Euler summability was first investigated systematically by KNOPP.[1] MEYER-KÖNIG[2] proved the following high-indices theorem for Euler summability: Assume that $a_k = 0$ except if

$$k = n_j \quad \text{where} \quad \frac{n_{j+1}}{n_j} \geq c > 1. \tag{1}$$

Then if $\sum_{k=0}^{\infty} a_k$ is Euler summable, it is convergent. MEYER-KÖNIG further conjectured that the theorem remains true if (1) is replaced by the much weaker condition $n_{j+1} - n_j > c\, n_j^{1/2}$ where $c > 0$ is any constant. It is not hard to see that MEYER-KÖNIG's conjecture if true is certainly best possible.

I succeeded in proving the following somewhat weaker theorem:[3] Let $\sum_{k=0}^{\infty} a_k$ be Euler summable, further $a_k = 0$ except if $k = n_j$ where $n_{j+1} - n_j > C n_j^{1/2}$ where C is a sufficiently large constant. Then $\sum_{k=0}^{\infty} a_k$ is convergent.

The main point in these theorems is that no restriction is placed on the speed with which a_k tends to infinity. As far as I know no analogous

[1] K. KNOPP, Über das Eulersche Summierungsverfahren. I, *Math. Zeitschr.*, **15** (1922), pp. 226—253; II, **18** (1923), pp. 125—156.

[2] W. MEYER-KÖNIG, Die Umkehrung des Euler—Knoppschen und des Borelschen Limitierungsverfahrens auf Grund einer Lückenbedingung, *Math. Zeitschr.*, **49** (1943—44), pp. 151—160.

[3] P. ERDŐS, *Acad. Serbe Sci. Publ. Inst. Math.*, **4** (1952), pp. 51—56. Recently MEYER-KÖNIG proved his conjecture: W. MEYER-KÖNIG, Bemerkung zu einem Lückenumkehrsatz von H. R. Pitt, *Math. Zeitschr.*, **57** (1952—53), pp. 351—352.

theorem is known for Borel summability. High-indices theorems have, in fact, been proved for Borel summability, e. g. Theorem of PITT[4] which will be used later in this paper, but as far as I know the growth of the a-s was always restricted.

The series $\sum_{k=0}^{\infty} a_k$ is said to be Borel summable to the sum s if

$$\lim_{x\to\infty} e^{-x} \sum_{k=0}^{\infty} s_k \frac{x^k}{k!} = s, \quad s_k = \sum_{i=0}^{k} a_i.$$

In this paper I prove the following

THEOREM. *Let* $\sum_{k=0}^{\infty} a_k$ *be Borel summable. Assume that* $a_k = 0$ *except if* $k = n_j$ *where*

$$n_{j+1} - n_j > c_1 n_j^{1/2} \tag{2}$$

($c_1 > 0$ *is any constant). Further let be*

$$\sum_{j=1}^{\infty} \frac{1}{n_{j+1} - n_j} < \infty. \tag{3}$$

Then $\sum_{k=0}^{\infty} a_k$ *is convergent.*

Throughout this paper $c_1, c_2, \ldots$ will denote positive absolute constants.

The proof of our Theorem will be fairly complicated. It could be somewhat simplified if we would replace (2) by the following condition: $n_{j+1} - n_j > C n_j^{1/2}$ where C is a sufficiently large constant. The somewhat large extra trouble in proving our Theorem might be justified by the possibility that our Theorem is best possible in the following sense: Let $n_1, n_2, \ldots$ be a sequence of integers which does not satisfy both (2) and (3). Then there exists a divergent series $\sum_{k=0}^{\infty} a_k$, $a_k = 0$ except if $k = n_j$, and $\sum_{k=0}^{\infty} a_k$ is Borel summable.

If $n_1 < n_2, \ldots$ does not satisfy (2), it is easy to construct such a series. Thus only the necessity of (3) is in doubt. In fact, it is quite possible that analogously to the MEYER-KÖNIG conjecture condition (3) is entirely superfluous. At present I am unable to decide these questions.

[4] H. R. PITT, General Tauberian theorems, *Proc. London Math. Soc.*, Ser. II, **44** (1938), pp. 243—288, Theorem 17.

One final remark: We might modify the definition of Borel summability as follows: $\sum_{k=0}^{\infty} a_k$ is summable B' if

$$\lim_{t\to\infty} e^{-t} \sum_{k=0}^{\infty} s_k \frac{t^k}{k!}$$

exists as t runs through the integers. It is not hard to show that if $n_1 < n_2, \ldots$ is any sequence of integers, there exists a divergent series $\sum_{k=0}^{\infty} a_k$ which is summable B' despite the fact that $a_k = 0$ except if $k = n_j$. Thus no high-indices theorem holds for B' summability unless we restrict the speed with which $s_k \to \infty$ (over and beyond the trivial restriction $s_k \frac{t^k}{k!} \to 0$ for every t).

LEMMA 1. *Let*

$$|s_n|^{1/n} = O(1) \qquad (n \to \infty),$$

further $a_k = 0$ *for* $k \neq n_j$ $(j = 1, 2, \ldots)$, $n_{j+1} - n_j > c_1 n_j^{1/2}$. *Then if* $\sum_{k=0}^{\infty} a_k$ *is Borel summable, then it is also convergent.*

This is a result of PITT.[4]

For the rest of this paper we can assume that for infinitely many n the inequality

$$|s_n| > K^n$$

holds, where K is an arbitrary constant; henceforth we shall assume that for infinitely many n

$$|s_n| > 100^n. \tag{4}$$

Let us denote by $f(x)$ the index of the maximal term of the series $e^{-x} \sum_{k=0}^{\infty} s_k \frac{t^k}{k!}$; if there are several such terms, $f(x)$ has the smallest possible value.

LEMMA 2. $f(x)$ *is a non-decreasing function of* x.

This obviously follows from the following statement: Let $y > x$, $k_2 > k_1$. Assume that

$$\frac{x^{k_2}}{k_2!} > \frac{x^{k_1}}{k_1!}, \tag{5}$$

then

$$\frac{y^{k_2}}{k_2!} > \frac{y^{k_1}}{k_1!}. \tag{6}$$

This is an immediate consequence of the identities

$$\frac{y^{k_2}}{k_2!}=\frac{x^{k_2}}{k_2!}\left(\frac{y}{x}\right)^{k_2}, \qquad \frac{y^{k_1}}{k_1!}=\frac{x^{k_1}}{k_1!}\left(\frac{y}{x}\right)^{k_1}$$

and of $k_2>k_1$.

Let

$$n_{i_1}, n_{i_2}, \ldots$$

be an infinite sequence of positive integers which satisfy (4) and for which

$$|s_{n_{i_l}}|>|s_m| \quad \text{for} \quad m<n_{i_l}. \tag{7}$$

Then we have

LEMMA 3.

$$f(n_{i_l})\geqq n_{i_l}.$$

Lemma 3 follows from (7).

LEMMA 4. *We have for* $n_{i_l}\leqq x\leqq 2n_{i_l}$

$$F(x)>30^{n_{i_l}}$$

where $F(x)$ *denotes the maximal term of the series* $e^{-x}\sum_{k=0}^{\infty}\frac{s_k}{k!}x^k$.

PROOF. We obtain from (4) by application of Stirling's formula

$$F(x)\geqq e^{-x}\frac{x^{n_{i_l}}}{n_{i_l}!}|s_{n_{i_l}}|>e^{-2n_{i_l}}\frac{n_{i_l}^{\,n_{i_l}}}{n_{i_l}!}100^{n_{i_l}}\geqq$$

$$\geqq e^{-2n_{i_l}}\frac{e^{n_{i_l}}}{\sqrt{2\pi n_{i_l}}}100^{n_{i_l}}=\left(\frac{100}{e}\right)^{n_{i_l}}\frac{1}{\sqrt{2\pi n_{i_l}}}>30^{n_{i_l}},$$

q. e. d.

We are going to prove that

$$\lim_{l\to\infty}\max_{n_{i_l}\leqq x\leqq 2n_{i_l}} e^{-x}\left|\sum_{k=0}^{\infty}\frac{s_k}{k!}x^k\right|=\infty. \tag{8}$$

Clearly, (8) implies that $\sum_{k=0}^{\infty}a_k$ cannot be Borel summable. Thus (8) implies our Theorem. Thus it will be sufficient to prove (8).

Before proving (8) we simplify our notation.

Denote $m_1=n_{i_l}$, $m_1<m_2<\cdots<m_r$ the n_i-s in the interval $(n_{i_l}, 2n_{i_l})$ (i. e. the m-s are the n_i-s in $(n_{i_l}, 2n_{i_l})$ for which a_k does not have to be 0).

It follows from the Lemmas 2 and 3 that for $x > m_1$ $f(x) \geqq m_1$. Now we distinguish two cases. In the first case for arbitrarily large K there exist infinitely many j-s for which either

$$f(x) = [x] \qquad (m_j \leqq y \leqq x \leqq y + Ky^{1/2} \leqq m_{j+1}) \tag{9}$$

or

$$f(x) = N \qquad (m_j \leqq y \leqq x \leqq y + Ky^{1/2} \leqq m_{j+1}); \tag{10}$$

it is easy to see that either $N = m_i$ or $N = m_{i+1} - 1$. $N = m_{i+1} - 1$ holds for $N \leqq y, N = m_i$ for $N \geqq y + Ky^{1/2}$. We may assume that y is an integer. The second case holds when for every number of the interval $(m_1, 2m_1)$ neither (9) nor (10) holds. We make use of (3) only in the second case.

Let us treat the first case. First we assume that (9) holds. We put $T = y + \left[\frac{K}{2} y^{1/2}\right]$, $L = y + [Ky^{1/2}]$ and show that

$$e^{-T} \left| \sum_{k=0}^{\infty} \frac{s_k}{k!} T_k \right| \to \infty \quad \text{for} \quad m_1 \to \infty. \tag{11}$$

We have

$$\sum_{k=0}^{\infty} \frac{s_k}{k!} T^k = \sum_{k=0}^{m_j - 1} + \sum_{k=m_j}^{m_{j+1}-1} + \sum_{k=m_{j+1}}^{\infty} = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3. \tag{12}$$

We estimate $\sum_2$ from below. We have from $s_{m_j} = s_{m_j+1} = \cdots = s_{m_{j+1}-1}$

$$\left|\sum\nolimits_2\right| = |s_{m_j}| \sum_{k=m_j}^{m_{j+1}-1} \frac{T^k}{k!} \geqq |s_{m_j}| \sum_{k=y}^{T} \frac{T^k}{k} \geqq$$

$$\geqq |s_{m_j}| \frac{T^y}{y!} \left[\frac{K}{2} y^{1/2}\right] = |s_{m_j}| \frac{y^y}{y!} \left(\frac{T}{y}\right)^y \left[\frac{K}{2} y^{1/2}\right].$$

We have by Stirling's formula

$$\frac{y^y}{y!} = (1 + o(1)) \frac{e^y}{\sqrt{2\pi y}} \qquad (y \to \infty),$$

further

$$\left(\frac{T}{y}\right)^y = \left(\frac{y + \left[\frac{K}{2} y^{1/2}\right]}{y}\right)^y = (1 + o(1))\, e^{\frac{K}{2} y^{1/2} - \frac{K^2}{8}}.$$

Hence

$$\left|\sum\nolimits_2\right| > |s_{m_j}| \frac{K}{2\sqrt{2\pi}} (1 + o(1))\, e^{y + \frac{K}{2} y^{1/2} - \frac{K^2}{8}},$$

i. e.

$$|\textstyle\sum_2| > |s_{m_j}| e^{T-\frac{K^2}{8}}. \tag{13}$$

Next we estimate $\sum_1$ and $\sum_3$ from above.

$$\sum\nolimits_1 = \sum_{k=0}^{m_j-1} \frac{s_k}{k!} T^k = \sum_{k=0}^{m_j-1} \frac{s_k}{k!} y^k \left(\frac{T}{y}\right)^k.$$

Since by (9)

$$\frac{|s_k|}{k!} y^k \leqq \frac{|s_y|}{y!} y^y = |s_{m_j}| \frac{y^y}{y!},$$

we have

$$|\textstyle\sum_1| \leqq |s_{m_j}| \dfrac{y^y}{y!} \left(\dfrac{T}{y}\right)^{m_j} \left(1 + \dfrac{y}{T} + \dfrac{y^2}{T^2} + \cdots\right) =$$

$$= |s_{m_j}| \frac{y^y}{y!} \left(\frac{T}{y}\right)^{m_j} \frac{T}{T-y} \leqq |s_{m_j}| \frac{y^y}{y!} \left(\frac{T}{y}\right)^{y} \frac{T}{T-y},$$

thus again, by applying Stirling's formula, $\left(\frac{T}{y}\right)^y = (1+o(1)) e^{\frac{K}{2}y^{1/2} - \frac{K^2}{8}}$ and $\frac{T}{T-y} < \frac{4}{K} y^{1/2}$ $\left(\text{this is a consequence of } T = y + \left[\frac{K}{2} y^{1/2}\right]\right)$

$$|\textstyle\sum_1| \leqq \dfrac{4}{K} |s_{m_j}| e^{T-\frac{K^2}{8}} \tag{14}$$

and by the same method we obtain

$$|\textstyle\sum_3| \leqq \dfrac{4}{K} |s_{m_j}| e^{T-\frac{K^2}{8}}. \tag{15}$$

From (13), (14) and (15) we obtain for sufficiently large but fixed K

$$e^{-T} \left| \sum_{k=0}^{\infty} \frac{s_k}{k!} T^k \right| > \frac{1}{2e^{K^2/8}} |s_{m_j}|, \tag{16}$$

thus because of Lemma 4 our Theorem is proved if (9) holds.

Next assume that (10) holds. Then either

$$N \leqq m_j - 1 \tag{17}$$

or

$$N \geqq m_{j+1} \tag{18}$$

where N is the number defined in (10); i. e. either $N = m_i$ or $N = m_{i+1} - 1$. First we assume that (17) holds, i. e. we have

$$f(x) = m_{i+1} - 1 \qquad (m_j \leqq y \leqq x \leqq y + Ky^{1/2} \leqq m_{j+1}) \tag{19}$$

where $i+1 \leqq j$. We put again $T=y+\left[\frac{K}{2}\,y^{1/2}\right]$, $L=y+[Ky^{1/2}]$ and show (11). We have

$$\sum_{k=0}^{\infty} \frac{s_k}{k!}\,T^k = \sum_{k=0}^{m_i-1} + \sum_{k=m_i}^{m_{i+1}-1} + \sum_{k=m_{i+1}}^{\infty} = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 .$$

First we estimate $\sum_2$ from below. We have

$$\left|\sum\nolimits_2\right| = \left|\sum_{k=m_i}^{m_{i+1}-1} \frac{s_k}{k!}\,T^k\right| =$$

$$= \frac{|sm_{i+1}-1|}{(m_{i+1}-1)!}\,T^{m_{i+1}-1}\left(1+\frac{m_{i+1}-1}{T}+\frac{(m_{i+1}-1)(m_{i+1}-2)}{T^2}+\cdots\right) >$$

$$> \frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}\,T^{m_{i+1}-1}\left(1+\frac{m_{i+1}-c_1 m_{i+1}^{1/2}}{T}+\cdots+\left(\frac{m_{i+1}-1-c_1 m_{i+1}^{1/2}}{T}\right)^{c_1 m_{i+1}^{1/2}}\right) =$$

$$= \frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}\,T^{m_{i+1}-1}\left(1-\left(\frac{m_{i+1}-c_1 m_{i+1}^{1/2}}{T}\right)^{c_1 m_{i+1}^{1/2}}\right)\frac{T}{T-m_{i+1}+c_1 m_{i+1}^{1/2}} .$$

Since (by $m_{i+1} \leqq m_j < T$)

$$\left(\frac{m_{i+1}-c_1 m_{i+1}^{1/2}}{T}\right)^{c_1 m_{i+1}^{1/2}} < \left(1-\frac{c_1}{m_{i+1}^{1/2}}\right)^{c_1 m_{i+1}^{1/2}} < e^{-c_1^2},$$

further for $K > 4c_1$

$$\frac{T}{T-m_{i+1}+c_1 m_{i+1}^{1/2}} > \frac{1}{2}\,\frac{T}{T-m_{i+1}} ,$$

we have

$$\left|\sum\nolimits_2\right| \geqq c_2\,\frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}\,T^{m_{i+1}-1}\,\frac{T}{T-m_{i+1}} , \tag{20}$$

where c_2 is a constant depending only on c_1 defined in (2).

We now estimate $\sum_1$ and $\sum_3$. We have by (7)

$$\left\{ \begin{aligned} \left|\sum\nolimits_1\right| = \left|\sum_{k=0}^{m_i-1} \frac{s_k}{k!}\,T^k\right| \leqq \sum_{k=0}^{m_1-1} \frac{|s_k|}{k!}\,T^k + \sum_{l=1}^{i-1} |s_{m_l}| \sum_{k=m_l}^{m_{l+1}-1} \frac{T^k}{k!} \leqq \\ \leqq |s_{m_1}| \sum_{k=0}^{m_1-1} \frac{T^k}{k!} + \sum_{l=1}^{i-1} |s_{m_l}| \sum_{k=m_l}^{m_{l+1}-1} \frac{T^k}{k!} , \end{aligned} \right. \tag{21}$$

further

$$\sum_{k=m_l}^{m_{l+1}-1} \frac{T^k}{k!} = \frac{T^{m_{l+1}-1}}{(m_{l+1}-1)!}\left(1+\frac{m_{l+1}-1}{T}+\cdots\right) < \frac{T^{m_{l+1}-1}}{(m_{l+1}-1)!}\,\frac{T}{T-m_{l+1}} .$$

Now since $f(y)=m_{i+1}-1$, we have (for formal reasons we will replace

s_{m_j} by $s_{m_{j+1}-1}$)

$$|s_{m_{l+1}-1}|\,\frac{T^{m_{l+1}-1}}{(m_{l+1}-1)!}=|s_{m_{l+1}-1}|\,\frac{y^{m_{l+1}-1}}{(m_{l+1}-1)!}\left(\frac{T}{y}\right)^{m_{l+1}-1}\leqq$$

$$\leqq\frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}\,y^{m_{i+1}-1}\left(\frac{T}{y}\right)^{m_{l+1}-1}=\frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}\,T^{m_{i+1}-1}\left(\frac{y}{T}\right)^{m_{i+1}-m_{l+1}}.$$

Hence

$$\left|\sum\nolimits_1\right|\leqq\frac{|s_{m_1}|}{m_1!}\,T^{m_1}\frac{T}{T-m_1}+\frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}\,T^{m_{i+1}-1}\sum_{l=1}^{i-1}\left(\frac{y}{T}\right)^{m_{i+1}-m_{l+1}}\frac{T}{T-m_{l+1}}\leqq$$

$$\leqq\frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}\,T^{m_{i+1}-1}\left\{\left(\frac{y}{T}\right)^{m_{i+1}-m_1}\frac{T}{T-m_1}+\sum_{l=1}^{i-1}\left(\frac{y}{T}\right)^{m_{i+1}-m_{l+1}}\frac{T}{T-m_{l+1}}\right\}$$

or

$$\left|\sum\nolimits_1\right|\leqq\frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}\,T^{m_{i+1}-1}\frac{T}{T-m_{i+1}}\cdot 2\sum_{l=1}^{i-1}\left(\frac{y}{T}\right)^{m_{i+1}-m_{l+1}}. \tag{22}$$

We show that the factor $\sum_{l=1}^{i-1}\left(\frac{y}{T}\right)^{m_{i+1}-m_{l+1}}$ is arbitrarily small if K of (10) is sufficiently large. We have

$$\sum_{l=1}^{i-1}\left(\frac{y}{T}\right)^{m_{i+1}-m_{l+1}}=\left(\frac{y}{T}\right)^{m_{i+1}-m_i}\sum_{l=1}^{i-1}\left(\frac{y}{T}\right)^{m_i-m_{l+1}};$$

since m_i and m_l lie in the interval $(m_1, 2m_1)$ and by (2) $m_{l+1}-m_l>c_1 m_l^{1/2}$, we obtain

$$m_i-m_{l+1}\geqq(i-(l+1))\frac{c_1}{2}\,y^{1/2}.$$

Hence

$$\sum_{l=1}^{i-1}\left(\frac{y}{T}\right)^{m_i-m_{l+1}}\leqq\sum_{r=0}^{\infty}\left(\frac{y}{T}\right)^{\frac{c_1 r}{2}y^{1/2}}\leqq 2\sum_{r=0}^{\infty}e^{-c_1\frac{k}{4}r}=\frac{2}{1-e^{c_1\frac{k}{4}}}.$$

Further we have

$$\left(\frac{y}{T}\right)^{m_{i+1}-m_i}<2\left(1+\frac{K}{2y^{1/2}}\right)^{-c_1 m_i^{1/2}}\leqq 2\left(1+\frac{K}{2y^{1/2}}\right)^{-\frac{c_1}{2}y^{1/2}}$$

which is arbitrarily small if K is large enough. Hence

$$\sum_{l=1}^{i-1}\left(\frac{y}{T}\right)^{m_{i+1}-m_{l+1}}<\varepsilon \tag{23}$$

if K is sufficiently large. We obtain from (20), (22) and (23)

$$\left|\sum\nolimits_1\right|<\frac{1}{10}\left|\sum\nolimits_2\right| \tag{24}$$

for sufficiently large K. Next we estimate $\sum_3$ from above.

$$\left|\sum\nolimits_3\right| = \left|\sum_{k=m_{j+1}}^{\infty} \frac{s_k}{k!} T^k\right| \leqq \sum_{l=i+1}^{\infty} |s_{m_l}| \sum_{k=m_l}^{m_{l+1}-1} \frac{T^k}{k!} =$$

$$= \sum_{l=i+1}^{j-1} |s_{m_l}| \sum_{k=m_l}^{m_{l+1}-1} \frac{T^k}{k!} + |s_{m_j}| \sum_{k=m_j}^{T-1} \frac{T^k}{k!} + |s_{m_j}| \sum_{k=T}^{m_{j+1}-1} \frac{T^k}{k!} +$$

$$+ \sum_{l=j+1}^{\infty} |s_{m_l}| \sum_{k=m_l}^{m_{l+1}-1} \frac{T^k}{k!} = \sum\nolimits_3' + \sum\nolimits_3'' + \sum\nolimits_3''' + \sum\nolimits_3''''.$$

We have as in the estimation of $\sum_1$

$$\sum\nolimits_3' = \sum_{l=i+1}^{j-1} \frac{|s_{m_{l+1}-1}|}{(m_{l+1}-1)!} T^{m_{l+1}-1} \left(1 + \frac{m_{l+1}-1}{T} + \frac{(m_{l+1}-1)(m_{l+1}-2)}{T^2} + \cdots\right) \leqq$$

$$\leqq \sum_{l=i+1}^{j-1} \frac{|s_{m_{l+1}-1}|}{(m_{l+1}-1)!} T^{m_{l+1}-1} \frac{T}{T-m_{l+1}} = \sum_{l=i+1}^{j-1} \frac{|s_{m_{l+1}-1}|}{(m_{l+1}-1)!} L^{m_{l+1}-1} \left(\frac{T}{L}\right)^{m_{l+1}-1} \frac{T}{T-m_{l+1}} \leqq$$

$$\leqq \frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!} T^{m_{i+1}-1} \frac{T}{T-m_{i+1}} \sum_{l=i+1}^{j-1} \left(\frac{T}{L}\right)^{m_{l+1}-m_{i+1}} \frac{T-m_{i+1}}{T-m_{l+1}},$$

i. e.

$$\sum\nolimits_3' \leqq \frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!} T^{m_{i+1}-1} \frac{T}{T-m_{i+1}} \sum_{l=i+1}^{j-1} \left(\frac{T}{L}\right)^{m_{l+1}-m_{i+1}} \frac{T-m_{i+1}}{T-m_{l+1}}. \tag{25}$$

It is easy to see that

$$\sum_{l=i+1}^{j-1} \left(\frac{T}{L}\right)^{m_{l+1}-m_{i+1}} \frac{T-m_{i+1}}{T-m_{l+1}} < \varepsilon, \tag{26}$$

if K is sufficiently large. Hence for sufficiently large K

$$\sum\nolimits_3' < \frac{1}{10} \left|\sum\nolimits_2\right|. \tag{27}$$

We have

$$\sum\nolimits_3'''' = \sum_{l=j+1}^{\infty} |s_{m_l}| \sum_{k=m_l}^{m_{l+1}-1} \frac{T^k}{k!} = \sum_{l=j+1}^{\infty} \frac{|s_{m_l}|}{m_l!} T^{m_l} \left(1 + \frac{T}{m_l+1} + \cdots\right) \leqq$$

$$\leqq \sum_{l=j+1}^{\infty} \frac{|s_{m_l}|}{m_l!} T^{m_l} \frac{m_l+1}{m_l+1-T} = \sum_{l=j+1}^{\infty} \frac{|s_{m_l}|}{m_l!} L^{m_l} \left(\frac{T}{L}\right)^{m_l} \frac{m_l+1}{m_l+1-T} \leqq$$

$$\leqq \frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!} T^{m_{i+1}-1} \sum_{l=j+1}^{\infty} \left(\frac{T}{L}\right)^{m_l-m_{i+1}+1} \frac{m_l+1}{m_l+1-T} =$$

$$= \frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!} T^{m_{i+1}-1} \frac{T}{T-m_{i+1}} \sum_{l=j+1}^{\infty} \left(\frac{T}{L}\right)^{m_l-m_{i+1}+1} \frac{(m_l+1)(T-m_{i+1})}{T(m_l+1-T)}.$$

It is again easy to see that for sufficiently large K

(28) $$\sum_{l=j+1}^{\infty}\left(\frac{T}{L}\right)^{m_l-m_{i+1}+1}\frac{(m_l+1)(T-m_{i+1})}{T(m_l+1-T)}<\varepsilon$$

and hence

(29) $$\sum\nolimits_3'''' < \frac{1}{10}\left|\sum\nolimits_2\right|.$$

It remains to estimate $\sum_3''$ and $\sum_3'''$. We have

$$\left|\sum\nolimits_3''\right| = |s_{m_j}|\sum_{k=m_j}^{T-1}\frac{T^k}{k!} = \frac{|s_T|}{T!}T^T\left(1+\frac{T-1}{T}+\cdots\right)\leqq$$

$$\leqq \frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}L^{m_{i+1}-1}\left(\frac{T}{L}\right)^T(T-m_j)=$$

$$=\frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}T^{m_{i+1}-1}\frac{T}{T-m_{i+1}}\left(\frac{T}{L}\right)^{T-m_{i+1}+1}\frac{(T-m_{i+1})(T-m_j)}{T}.$$

Since for sufficiently large K

$$\left(\frac{T}{L}\right)^{T-m_{i+1}+1}\frac{(T-m_{i+1})(T-m_j)}{T}<\varepsilon,$$

we have for sufficiently large K

(30) $$\sum\nolimits_3'' \leqq \frac{1}{10}\left|\sum\nolimits_2\right|$$

and similarly

(31) $$\sum\nolimits_3''' \leqq \frac{1}{10}\left|\sum\nolimits_2\right|.$$

It follows from (20), (24), (27), (29), (30) and (31)

(32) $$e^{-T}\left|\sum_{k=0}^{\infty}\frac{s_k}{k!}T^k\right| > c_3\frac{|s_{m_{i+1}-1}|}{(m_{i+1}-1)!}T^{m_{i+1}-1}\frac{T}{T-m_{i+1}}.$$

Hence, by Lemma 4, (11) follows. If, instead of (17), (18) holds, (11) can be proved similarly and we omit the details. Thus our Theorem is proved in the first case.

Let us now treat the second case. First it is obvious that the number of the m_j-s in $(m_1, 2m_1)$ is $o(m_1^{1/2})$. This is an immediate consequence of (3). Hence it follows that the sum of the length of the intervals (m_j, m_{j+1}) in $(m_1, 2m_1)$ with $m_{j+1}-m_j<Km_j^{1/2}$ is $o(m_1)$.

We now split the numbers x in $(m_1, 2m_1)$ into three classes. In the first class are the numbers x with $f(x)=[x]$, in the second the x with $f(x)<x$, in the third the x with $f(x)>x$.

First we show that the sum of the length of the intervals of the first class is $o(m_1)$. It clearly suffices to consider the intervals (m_j, m_{j+1}) satisfying $m_{j+1} - m_j > K m_j^{1/2}$. Now we observe that the numbers x $(m_j \leqq x \leqq m_{j+1})$ for which $f(x) = [x]$ form a single interval; (this is clear since $f(x)$ is monotonic and if $f(x) \neq [x]$ then either $f(x) = m_i$ or $f(x) = m_{i+1} - 1$). Now since the number of m_j-s in $(m_1, 2m_1)$ is $o(m_1^{1/2})$ and we are in the second case, the result follows. Observing that by (7) $f(m_1) \geqq m_1$, we see by similar arguments that the sum of the intervals of the second class is also $o(m_1)$. Thus it follows that the sum of the length of the intervals in the third class is greater than $(1-\varepsilon)m_1$.

If $f(x) > [x]$, we have $f(x) = n_j$ $(> m_1)$ for some j. Let us denote by (α_j, β_j) the interval for which $f(x) = n_j$ $(\alpha_j \leqq x \leqq \beta_j)$. We have (as the length of the intervals in the third class is greater than $(1-\varepsilon)m_1$)

$$\sum_{m_1 \leqq \alpha_j \leqq \beta_j \leqq 2m_1} (\beta_j - \alpha_j) > (1-\varepsilon) m_1 \tag{33}$$

($\varepsilon > 0$ but arbitrarily small). As (10) does not hold, we have

$$\sum_{m_1 < n_j < 4m_1} (\beta_j - \alpha_j) = o(m_1), \tag{34}$$

hence

$$\sum_{n_j > 4m_1} (\beta_j - \alpha_j) > (1-\varepsilon) m_1 . \tag{35}$$

We may assume without loss of generality that between k^3 and $(k+1)^3$ there lies at least one n_j; we can assure this by introducing besides the old n_j-s new ones. The enlarged sequence obviously satisfies (2) and (3).

Next we prove

LEMMA 5.[5] *Denote by* (α_j, β_j) *the interval for which*

$$f(x) = n_j \quad (\alpha_j \leqq x \leqq \beta_j) \quad (n_j > 4m_1).$$

Let K *be an arbitrarily large constant. Then for sufficiently large* m_1 *there exist a* j *for which*

$$\frac{\beta_j}{\alpha_j} > 1 + \max_{t \neq 0} \frac{K|t|^{1/2}}{|n_{j+t} - n_j|}.$$

Clearly, Lemma 5 got considerably strengthened by the assumption that between k^3 and $(k+1)^3$ there lies at least one n_j.

[5] A similar lemma is used in a paper by MACINTYRE and myself, *Edinburgh Math. Proc.*, Ser. 2, **10** (1954).

Proof. Assume Lemma 5 is false. Then we have for every j

$$(\beta_j-\alpha_j)<K\alpha_j \max_{t\neq 0}\frac{|t|^{1/2}}{|n_{j+t}-n_j|}.$$

It follows from $\alpha_j<2m_1$ that

$$\sum_{n_j>4m_1}(\beta_j-\alpha_j)<2Km_1\sum_{n_j>4m_1}\max_{t\neq 0}\frac{|t|^{1/2}}{|n_{j+t}-n_j|}; \tag{36}$$

we have by the inequality of the arithmetic and harmonic means for every positive integer t

$$\frac{t^{1/2}}{n_{j+t}-n_j}=\frac{1}{t^{1/2}}\frac{t}{\sum_{k=j}^{t+j-1}(n_{k+1}-n_k)}\leqq\frac{1}{t^{3/2}}\sum_{k=j}^{j+t-1}\frac{1}{n_{k+1}-n_k}$$

and for every negative integer t similarly

$$\frac{|t^{1/2}|}{n_j-n_{j+t}}\leqq\frac{1}{|t|^{3/2}}\sum_{k=j+t}^{j}\frac{1}{n_{k+1}-n_k}.$$

t_j denotes the integer for which

$$\max_{t\neq 0}\frac{|t|^{1/2}}{|n_{j+t}-n_j|}=\frac{|t_j|^{1/2}}{|n_{j+t_j}-n_j|}.$$

Then we have for $t_j\geqq -M$ (where M is an arbitrarily large but fixed integer)

$$\sum_{\substack{n_j>4m_1\\ t_j\geqq -M}}\frac{|t_j^{1/2}|}{|n_{j+t_j}-n_j|}\leqq 2\sum_{t=1}^{\infty}t^{-3/2}\sum_{n_j\geqq n_{\lambda-M}}\frac{1}{n_{j+1}-n_j},$$

where n_λ denotes the smallest n_i with $n_i\geqq 4m_1$ and

$$\sum_{\substack{n_j>4m_1\\ t_j<-M}}\frac{|t_j|^{1/2}}{|n_{j+t_j}-n_j|}<2\sum_{t>M}t^{-3/2}\sum_{j=1}^{\infty}\frac{1}{n_{j+1}-n_j}.$$

Therefore by (3) and (36)

$$\sum_{n_j>4m_1}(\beta_j-\alpha_j)<2K\varepsilon m_1$$

for an arbitrarily small $\varepsilon>0$ if m_1 is sufficiently large. But this contradicts (35) if $\varepsilon<\frac{1}{4K}$. Thus our Lemma 5 is proved.

Let j be a number satisfying Lemma 5. Put

$$S=1+K\max_{t\neq 0}\frac{|t^{1/2}|}{|n_{j+t}-n_j|}.$$

Then there exist three numbers y_1, T, y_2 with

$$Sy_1 = T, \qquad ST = y_2$$

for which

$$f(y_1) = f(T) = f(y_2) = n_j.$$

We prove (11) in this case, too. We put again

$$\sum_{k=0}^{\infty} \frac{s_k}{k!} T^k = \sum_{k=0}^{n_j - 1} + \sum_{k=n_j}^{n_{j+1}-1} + \sum_{k=n_{j+1}}^{\infty} = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

First we estimate $\sum_2$ from below. Clearly

$$|\sum\nolimits_2| > |s_{n_j}| \frac{T^{n_j}}{n_j!}. \tag{37}$$

We now estimate $\sum_1$ and $\sum_3$ from above. We have

$$\begin{cases} |\sum_3| \leqq \sum\limits_{k=n_{j+1}}^{\infty} \dfrac{|s_k|}{k!} T^k = \sum\limits_{l=j+1}^{\infty} \dfrac{|s_{n_l}|}{n_l!} T^{n_l} \left(1 + \dfrac{T}{n_l+1} + \cdots\right) \leqq \\ \qquad \leqq \sum\limits_{l=j+1}^{\infty} \dfrac{|s_{n_l}|}{n_l!} T^{n_l} \dfrac{n_l}{n_l - T} \leqq 2 \sum\limits_{l=j+1}^{\infty} \dfrac{|s_{n_l}|}{n_l!} T^{n_l}, \end{cases} \tag{38}$$

since $n_l > 4m_1$, $T \leqq 2m_1$.

Further, by the definition of n_j,

$$\frac{|s_{n_l}|}{n_l!} T^{n_l} = \frac{|s_{n_l}|}{n_l!} y_2^{n_l} \left(\frac{T}{y_2}\right)^{n_l} \leqq \frac{|s_{n_j}|}{n_j!} y_2^{n_j} \left(\frac{T}{y_2}\right)^{n_l} = \frac{|s_{n_j}|}{n_j!} T^{n_j} \left(\frac{T}{y_2}\right)^{n_l - n_j}.$$

Hence from (38)

$$|\sum\nolimits_3| < 2 \frac{|s_{n_j}|}{n_j!} T^{n_j} \sum_{l=j+1}^{\infty} \left(\frac{T}{y_2}\right)^{n_l - n_j}. \tag{39}$$

We have for sufficiently large m_1

$$\left(\frac{T}{y_2}\right)^{n_l - n_j} < \left(1 + \frac{K(l-j)^{1/2}}{n_l - n_j}\right)^{-(n_l - n_j)} < 2e^{-K(l-j)^{1/2}}.[6]$$

Hence by (2)

$$\sum_{l=j+1}^{\infty} \left(\frac{T}{y_2}\right)^{n_l - n_j} < 2 \sum_{l=j+1}^{\infty} e^{-K(l-j)^{1/2}}$$

which is arbitrarily small if K is sufficiently large. Hence from (39) we

[6] This can be proved as follows: We have for $n > \frac{x^2}{2 \log 2}$ $\left(1 + \frac{x}{n}\right)^n > \frac{1}{2} e^x$ (namely, $n \log \left(1 + \frac{x}{n}\right) > x - \frac{x^2}{2n} > x - \log 2$). Now we have by (2) and (3) $l - j = o(n_l - n_j)$. Putting $x = K(l-j)^{1/2}$, $n = n_l - n_j$, the result follows.

have for sufficiently large K

$$|\textstyle\sum_3| \leqq \frac{1}{4} |\sum_2|. \tag{40}$$

Finally, we estimate $\sum_1$. Let us define the number λ by

$$n_\lambda \leqq T < n_{\lambda+1}.$$

Then we write

$$\sum_1 = \sum_{k=0}^{n_\lambda-1} + \sum_{k=n_\lambda}^{n_{\lambda+1}-1} + \sum_{k=n_{\lambda+1}}^{\infty} = \sum_1' + \sum_2'' + \sum_3'''.$$

We have

$$\begin{aligned} |\textstyle\sum_1'| &\leqq \sum_{l=1}^{\lambda-1} \frac{|s_{n_{l+1}-1}|}{(n_{l+1}-1)!} T^{n_{l+1}-1}\left(1+\frac{n_{l+1}-1}{T}+\cdots\right) \leqq \\ &\leqq \sum_{l=1}^{\lambda-1} \frac{|s_{n_{l+1}-1}|}{(n_{l+1}-1)!} T^{n_{l+1}-1} \frac{T}{T-n_{l+1}+1}. \end{aligned} \tag{41}$$

Now we have as before

$$|s_{n_{l+1}-1}| \frac{T^{n_{l+1}-1}}{(n_{l+1}-1)!} \leqq \frac{|s_{n_j}|}{n_j!} T^{n_j} \left(\frac{y_1}{T}\right)^{n_j-n_{l+1}+1};$$

hence by (41)

$$|\textstyle\sum_1'| \leqq \frac{|s_{n_j}|}{n_j!} T^{n_j} T \sum_{l=1}^{\lambda-1} \left(\frac{y_1}{T}\right)^{n_j-n_{l+1}+1}. \tag{42}$$

We have for sufficiently large m_1 as in the estimation of $\sum_3$

$$\left(1+\frac{K(j-l-1)^{1/2}}{n_j-n_{l+1}}\right)^{-(n_j-n_{l+1})} < 2e^{-K(j-l-1)^{1/2}},$$

i. e.

$$\left\{\begin{aligned} &\left(\frac{y_1}{T}\right)^{n_j-n_{l+1}} = \left(1+K\max_{t\neq 0}\frac{|t|^{1/2}}{|n_{j+t}-n_j|}\right)^{n_j-n_{l+1}} \leqq \\ &\leqq \left(1+\frac{K(j-l-1)^{1/2}}{n_j-n_{l+1}}\right)^{n_j-n_{l+1}} < 2e^{-K(j-l-1)^{1/2}} \leqq 2e^{-\frac{K}{2}\{(j-\lambda)^{1/2}+(\lambda-l-1)^{1/2}\}}. \end{aligned}\right. \tag{43}$$

Since between k^3 and $(k+1)^3$ there is at least one n_l, we have $j-\lambda > n_j^{1/3} - n_\lambda^{1/3}$ and since $n_j > 4m_1$, $n_\lambda < 2m_1$, we have

$$(j-\lambda)^{1/2} > ((4m_1)^{1/3}-(2m_1)^{1/3})^{1/2} = 2^{1/6}(2^{1/3}-1)^{1/2} m_1^{1/6} = c_4 m_1^{1/6},$$

hence by (42)

$$|\textstyle\sum_1'| \leqq 4\frac{|s_{n_j}|}{n_j!} T^{n_j} m_1 e^{-\frac{K}{2}c_4 m_1^{1/6}} \sum_{l=1}^{\lambda-1} e^{-\frac{K}{2}(\lambda-l-1)^{1/2}},$$

or since

$$\sum_{l=1}^{\lambda-1} e^{-\frac{K}{2}(\lambda-l-1)^{1/2}} < \sum_{l=1}^{\infty} e^{-\frac{K}{2}e^{1/2}} = O(1)$$

and $m_1 e^{-\frac{K}{2}c_4 m_1^{1/6}}$ is arbitrarily small if m_1 is sufficiently large, we have

$$(44) \qquad |\textstyle\sum_1'| < \frac{1}{12} |\sum_2|.$$

The estimation of $\sum_1''$ can be done similarly and we obtain for sufficiently large K

$$(45) \qquad |\textstyle\sum_1''| < \frac{1}{12} |\sum_2|.$$

Finally, we estimate $\sum_2'''$ from above. We have

$$(46) \qquad |\textstyle\sum_1'''| \leqq \displaystyle\sum_{k=n_{\lambda+1}}^{n_j-1} \frac{|s_k|}{k!} T^k \leqq \sum_{l=\lambda+1}^{j-1} \frac{|s_{n_l}|}{n_l!} T^{n_l} \left(1 + \frac{T}{n_l+1} + \cdots\right) = \sum_{l=\lambda+1}^{j-1} \frac{|s_{n_l}|}{n_l!} T^{n_l} \frac{n_l+1}{n_l+1-T}.$$

Further, we obtain as before

$$(47) \qquad \frac{|s_{n_l}|}{n_l!} T^{n_l} \leqq \frac{|s_{n_l}|}{n_j!} T^{n_j} \left(\frac{y_1}{T}\right)^{n_j-n_l};$$

taking into account that by the definition of y_1 and T

$$\left(\frac{y_1}{T}\right) = \left(1 + \max_{t \neq 0} \frac{K|t^{1/2}|}{|n_{j+t} - n_j|}\right)^{-1}$$

we have

$$(48) \qquad \left(\frac{y_1}{T}\right)^{n_j-n_l} < \left(1 + \frac{K(j-l)^{1/2}}{n_j - n_l)}\right)^{-(n_j-n_l)} < 2e^{-K(j-l)^{1/2}},$$

we have by (46)

$$(49) \qquad |\textstyle\sum_1'''| \leqq \frac{|s_{n_j}|}{n_j!} T^{n_j} 2 \displaystyle\sum_{l=\lambda+1}^{j-1} e^{-K(j-l)^{1/2}} \frac{n_l+1}{n_l+1-T}.$$

Consider the sum $\sum_{l=\lambda+1}^{j-1} e^{-K(j-l)^{1/2}} \frac{n_l+1}{n_l+1-T}$. Let λ' denote the greatest number for which $n_{\lambda'} - T \leqq \frac{n_{\lambda'}}{3}$, i. e. $n_{\lambda'} \leqq \frac{3}{2} T \leqq 3m_1$. Then write

$$\sum_{l=\lambda+1}^{j-1} = \sum_{l=\lambda+1}^{\lambda'} + \sum_{l=\lambda'+1}^{j-1}.$$

First we have

$$\sum_{l=\lambda+1}^{\lambda'} e^{-K(j-l)^{1/2}} \frac{n_l+1}{n_l+1-T} < 3m_1 \sum_{l=\lambda+1}^{\lambda'} e^{-\frac{K}{2}\{(j-\lambda')^{1/2}+(\lambda'-l)^{1/2}\}};$$

hence because of $n_j > 4m_1$, $n_{\lambda'} \leqq 3m_1$ and because of the fact that between k^3 and $(k+1)^3$ there lies at least one n_l we have as before

$$\text{(50)} \qquad \sum_{l=\lambda+1}^{\lambda'} e^{-K(j-l)^{1/2}} \frac{n_l+1}{n_l+1-T} \leqq m_1 e^{-\frac{K}{2}c_3 m_1^{1/6}} \sum_{l=1}^{\infty} e^{-\frac{K}{2}l^{1/2}} = o(1) \qquad (m_1 \to \infty).$$

Further we have

$$\sum_{l=\lambda'+1}^{j-1} e^{-K(j-l)^{1/2}} \frac{n_l+1}{n_l+1-T} < 3 \sum_{l=\lambda'+1}^{j-1} e^{-K(j-l)^{1/2}} < 3 \sum_{l=1}^{\infty} e^{-Kl^{1/2}} < 3 \int_0^{\infty} e^{-Kx^{1/2}} dx = \frac{6}{K^2},$$

i. e. for arbitrarily small $\varepsilon > 0$

$$\text{(51)} \qquad \sum_{l=\lambda'+1}^{j-1} e^{-K(j-l)^{1/2}} \frac{n_l+1}{n_l+1-T} < \varepsilon,$$

if K is sufficiently large.

We obtain from (46), (47), (49), (50) and (51) for sufficiently large K

$$\text{(52)} \qquad \left|\sum\nolimits_1'''\right| < \frac{1}{12}\left|\sum\nolimits_2\right|,$$

hence

$$\left|\sum_{k=0}^{\infty} \frac{s_k}{k!} T^k\right| > \left|\sum\nolimits_2\right| - \left|\sum\nolimits_1\right| - \left|\sum\nolimits_3\right| > \frac{1}{2}\left|\sum\nolimits_2\right|,$$

hence, by Lemma 4, (11) follows and the proof of our Theorem is complete.

We can generalize Borel summability as follows: Let $f(z) = \sum_{k=0}^{\infty} b_k z^k$ be an entire function, b_k real and $f(z) \to \infty$ for $z \to \infty$ along the positive axis. The series $\sum_{k=0}^{\infty} a_k$ is said to be summable f to s, if

$$\lim_{z\to\infty} \frac{1}{f(z)} \sum_{k=0}^{\infty} s_k b_k z^k = s, \qquad s_k = \sum_{l=0}^{k} a_l.$$

It seems that the following high-indices theorem holds for this summability method. There exists an increasing function $g(x)$ depending only on $f(z)$ so that if $\sum_{k=0}^{\infty} a_k$ is summable f and $a_k = 0$ except if $k = n_j$ where $n_{j+1} > g(n_j)$, then $\sum_{k=0}^{\infty} a_k$ converges.

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