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FUNCTIONS WHICH ARE SYMMETRIC ABOUT SEVERAL POINTS

BY

PAUL ERDÖS and MICHAEL GOLOMB
(Notre Dame University) (Purdue University)

1. Let $f(t)$ be a real-valued function. R. P. BOAS [1]) called $f(t)$ odd about the point $(x, f(x))$ if for all t except possibly a set of measure 0

$$f(x+t)+f(x-t)-2f(x)=0 \tag{1}$$

If $f(t)$ is odd about several points $(x_a, f(x_a))$ it is to be understood that the exceptional set may depend on x_a.

BOAS [1]) proves among others that if $f(t)$ is periodic, bounded on a set of positive measure, and satisfies (1) for a set of x's having positive measure then $f(t)$ is equivalent to a constant (i.e. $f(t)$ is constant almost everywhere). He also shows there exists a bounded periodic function not equivalent to a constant which is odd about a denumerable set of points. He proposes the question if a bounded periodic $f(t)$ exists not equivalent to a constant which is odd about a noncountable set of points. He remarks that it is clearly necessary to put on $f(t)$ some restriction like boundedness since every additive function (that is, every solution of the functional equation $f(x+y)=f(x)+f(y)$) satisfies (1) for all x.

We shall prove that such functions do exist. We shall also consider the more general functional equation

$$\Sigma_{m=1}^{n} \gamma_m f(z+c_m u)-f(z)=0 \tag{2}$$

where the γ_m, c_m are given complex numbers, $\Sigma_m \gamma_m = 1$, $c_m \neq 0$. We shall prove that if $f(u)$ is an essentially bounded complex-valued function of the complex variable u, not equivalent to a constant,

[1]) R. P. BOAS, Functions which are odd about several points, Nieuw Archief voor Wiskunde (1) 3, 27—32 (1953)

which satisfies (2) identically in u for at least one value of the complex variable z then

$$\text{g.l.b.}\left|\sum_{m=1}^{n}\gamma_m\,|c_m|^{ir}\left(\frac{c_m}{|c_m|}\right)^s\right| = 0 \qquad (-\infty < r, s < \infty) \quad (3)$$

Conversely, if this condition is satisfied then (2) has a bounded solution, not equivalent to a constant, for a set of z's of power c that is everywhere dense in the z plane.

2. Theorem 1. *There exists a bounded function $f(t)$, not equivalent to a constant, which satisfies* (1) *identically in t for a non-measurable set of x's which is everywhere dense and of power c.*

Let H be a Hamel basis of the real numbers, x_0 an arbitrary element of H, and put $H_- = H - \{x_0\}$. Every real number t can be written, in a unique way, as a finite sum $t = h_0 x_0 + \sum h_k x_{a_k}$, where $x_{a_k} \in H_-$ and the h's are rational numbers, $h_k \neq 0$. We shall write $h_0 = h_0(t)$ for the coefficient of x_0 in this decomposition of t. Then $h_0(r_1 t_1 + r_2 t_2) = r_1 h_0(t_1) + r_2 h_0(t_2)$ for any real t_1, t_2 and rational r_1, r_2.

Obviously, the set X_0 of real numbers x for which $h_0(x) = 0$ is non-measurable, of the power c, and everywhere dense. We use X_0 as the set of x's for which (1) is satisfied. We define $f(t)$ as follows:

$$f(t) = 0 \qquad \text{if } t \in X_0 \quad (4)$$
$$= \frac{h_0(t)}{|h_0(t)|} \quad \text{if } t \notin X_0$$

Clearly, $f(t)$ is bounded and not equivalent to a constant. Equation (1) is satisfied identically in t for all $x \in X_0$. To see this we have only to remark that if $t \in X_0$ then $(x \pm t) \in X_0$ and all summands in (1) vanish. If $t \notin X_0$ then $(x \pm t) \notin X_0$ and $h_0(x \pm t) = \pm h_0(t)$, hence $f(x + t) = - f(x - t)$, and (1) holds again.

We remark that $f(t)$ is periodic. Every $x \in X_0$ is a period.

3. We now turn to equation (2). We first prove that the existence of a bounded solution implies that condition (3) holds.

Theorem 2. *If $f(u)$ is essentially bounded, not equivalent to a constant, and satisfies* (2) *for almost all u and at least one value of z then* (3) *holds.*

It is no restriction to assume (2) is satisfied for $z_0 = 0$ since we may otherwise use $z^* = z - z_0$, $f^*(u) = f(u + z_0)$. Also we may assume $f(0) = 0$ since if $f(u)$ satisfies (2) so does $f(u) - f(0)$. We then have

$$\Sigma_{m=1}^{n} \gamma_m f(c_m u) = 0 \tag{5}$$

for almost all u. Putting $c_m = e^{a_m + ib_m}$, $u = e^{x+iy}$ (a_m, b_m, x, y real), $f(e^{x+iy}) = g(x, y)$, equation (5) becomes

$$\Sigma_m \gamma_m g(x + a_m, y + b_m) = 0 \tag{6}$$

This equation is satisfied for almost all x, y by the essentially bounded function $g(x, y)$.

Now let $A_1, \ldots, A_K$ be a rational basis for the numbers $a_1, \ldots, a_n$, and $B_1, \ldots, B_L$ a rational basis for $b_1, \ldots, b_n$. That is, $A_1, \ldots, A_K$ and $B_1, \ldots, B_L$ are linearly independent over the rationals, and there are integers a_{mk}, b_{ml} such that

$$a_m = \Sigma_{k=1}^{K} a_{mk} A_k,\ b_m = \Sigma_{l=1}^{L} b_{ml} B_l \ \ (m = 1, \ldots, n)$$

If $a_1 = a_2 = \ldots = a_n = 0$ or $b_1 = b_2 = \ldots = b_n = 0$ we take the corresponding basis to be vacuous.

Let α_k, β_l ($k = 1, \ldots, K$; $l = 1, \ldots, L$) be any integers. Equation (6) implies

$$\Sigma_m \gamma_m g(x + a_m + \Sigma_k \alpha_k A_k, y + b_m + \Sigma_l \beta_l B_l) = 0 \tag{7}$$

for almost all x, y. Let x, y be chosen such that (7) holds and such that the set of numbers $g(x + a_m + \Sigma_k \alpha_k A_k,\ y + b_m + \Sigma_l \beta_l B_l)$ ($\alpha_k, \beta_l = 0, \pm 1, \pm 2, \ldots; m = 1, \ldots, n$) is bounded. With these numbers as coefficients we form the Fourier series [2])

$$\Sigma_{(\alpha,\beta)} g(x + a_m + \Sigma_k \alpha_k A_k, y + b_m + \Sigma_l \beta_l B_l) \,.\, \exp(i\Sigma_k \alpha_k A_k r_k + i\Sigma_l \beta_l B_l s_l) \tag{8}$$

It represents a SCHWARTZ' distribution [3]) on the torus T^{K+L} whose points are the $(K + L)$-tuples $(r_1, \ldots, r_K, s_1, \ldots, s_L)$, where r_k, s_l are real numbers modulo $2\pi A_k^{-1}$, $2\pi B_l^{-1}$. By (7) we have

$$\Sigma_m \Sigma_{(\alpha,\beta)} \gamma_m g(x + a_m + \Sigma_k \alpha_k A_k, y + b_m + \Sigma_l \beta_l B_l) \,.\, \exp(i \Sigma_k \alpha_k A_k r_k + i \Sigma_l \beta_l B_l s_l) = 0,$$

or after renaming the α's, β's

$$\Sigma_{(\alpha,\beta)}\ g(x + \Sigma_k \alpha_k A_k, y + \Sigma_l \beta_l B_l) \exp(i \Sigma_k \alpha_k A_k r_k + i \Sigma_l \beta_l B_l s_l) \,.\, \Sigma_m \gamma_m \exp(- i \Sigma_k a_{mk} A_k r_k - i \Sigma_l b_{ml} B_l s_l) = 0. \tag{9}$$

[2]) Summation to be done over all integral values for $\alpha_1, \ldots, \alpha_K, \beta_1, \ldots, \beta_L$.

[3]) L. SCHWARTZ, Théorie des Distributions, Hermann and Cie, Paris 1951, tome II, Ch. VII, 1. — The theory of distributions is used here only to avoid somewhat lengthier, but probably more familiar arguments on weak convergence.

Now assume condition (3) is not satisfied. That is, there exists a number $\varepsilon > 0$ such that

$$| \Sigma_m \gamma_m \exp(- ia_m r - ib_m s) | \geq \varepsilon \qquad (10)$$

for all real r, s. Because of the rational independence of the numbers $A_1, \ldots, A_K$ and $B_1, \ldots, B_L$ the points $(A_1 r, \ldots, A_K r, B_1 s, \ldots, B_L s)$ $(-\infty < r, s < \infty)$ are everywhere dense on T^{K+L}. It follows that

$$| \Sigma_m \gamma_m \exp(- i \Sigma_k a_{mk} A_k r_k - i \Sigma_l b_{ml} B_l s_l | \geq \varepsilon$$

on T^{K+L}. Therefore, (9) implies that Fourier series (8) represents the zero distribution on T^{K+L}, which is possible only if all the coefficients in the series are zero. Thus, $g(x, y) = 0$ for almost all x, y. This is contrary to the hypotheses of the theorem. It follows that (10) cannot hold.

The proof shows that the boundedness of the numbers

$$g(x_0 + \Sigma_k \alpha_k A_k, y_0 + \Sigma_l \beta_l B_l),$$

assuming that they do not all vanish, for some fixed x_0, y_0 and α_k, $\beta_l = 0, \pm 1, \pm 2, \ldots$, already implies condition (3). We may also replace essential boundedness of $g(x, y)$ by the condition of essential boundedness of $(1 + x^2 + y^2)^{-N} g(x, y)$ for some sufficiently large N. Then, as in the case of bounded $g(x, y)$, Fourier series (8) converges to a distribution [4]).

4. The solution of equation (2) is, like that of equation (1), constructed with the aid of a HAMEL basis. Let H be such a basis of the complex numbers over the field $C(c_1, c_2, \ldots, c_n)$, which is the field resulting from adjoining the numbers $c_1, c_2, \ldots, c_n$ to the rational real numbers. Let z_0 be an arbitrary element of H, and put $H_- = H - \{z_0\}$. Every complex number u can be written, in a unique way, as a finite sum $u = h_0 z_0 + \Sigma h_k z_{\alpha_k}$, where $z_{\alpha_k} \in H_-$ and the h's are rational functions with rational real coefficients of $c_1, c_2, \ldots, c_n$; $h_k \neq 0$. We write again $h_0 = h_0(u)$ for the coefficient of z_0 in this decomposition. The set Z_0 of complex numbers z for which $h_0(z) = 0$ is non-measurable, of the power c, and everywhere dense. Of the same character is $\tilde{Z}_0$, the complementary set.

In the following we write, as before, $c_m = e^{a_m + ib_m}$ (a_m, b_m real). The lemma to be proved is independent of condition (3).

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[4]) L.c. Footnote 3.

Lemma. *For every pair of real numbers* r, s *there exists a function* $f(u)$ *such that*

$$(a) \quad f(u) = 0 \text{ on } Z_0, \qquad |f(u)| = 1 \text{ on } \tilde{Z}_0$$
$$(b) \quad |\Sigma_m \gamma_m f(z + c_m u) - f(z)| = 0 \qquad \text{if } u \in Z_0$$
$$= |\Sigma_m \gamma_m e^{i(a_m r + b_m s)}| \text{ if } u \in \tilde{Z}_0$$

for all $z \in Z_0$.

Using the function $h_0(u)$ defined above put $h_0(u) = e^{a_0(u) + ib_0(u)}$ ($a_0(u)$, $b_0(u)$ real). Then define

$$f(u) = 0 \qquad \text{if } u \in Z_0 \tag{11}$$
$$= e^{i(a_0(u)r + b_0(u)s)} \qquad \text{if } u \in \tilde{Z}_0$$

This function obviously satisfies assertion (*a*) of the Lemma as well as the first part of (*b*). If $z \in Z_0$, $u \in \tilde{Z}_0$ then $h_0(z + c_m u) = c_m h_0(u) = e^{a_m + a_0(u) + i(b_m + b_0(u))}$; thus $f(z + c_m u) = e^{i(a_m r + b_m s)} e^{i(a_0(u)r + b_0(u)s)}$, while $f(z) = 0$. This proves assertion (*b*) of the Lemma.

In this lemma, as in the following theorem, no use is made of the condition $\Sigma_m \gamma_m = 1$.

If there are numbers r, s for which

$$\Sigma_m \gamma_m |c_m|^{ir} \left(\frac{c_m}{|c_m|} \right)^s = 0 \tag{12}$$

then the function $f(u)$ of the Lemma, constructed for these numbers r, s, is a solution of (2). We prove a stronger result, the exact converse of Theorem 2.

Theorem 3. *If condition* (3) *holds then there exists a bounded function* $f(u)$, $f(u) = 0$ *on* Z_0, $|f(u)| = 1$ *on* $\tilde{Z}_0$, *which satisfies* (2) *identically in* u *for all* $z \in Z_0$.

Let r_k, s_k ($k = 1, 2, \dots$) be so chosen that

$$|\Sigma_m \gamma_m e^{i(a_m r_k + b_m s_k)}| < \frac{1}{k}.$$

Define the sequence of functions

$$f_k(u) = 0 \qquad \text{if } u \in Z_0$$
$$= e^{i(a_0(u) r_k + b_0(u) s_k)} \qquad \text{if } u \in \tilde{Z}_0.$$

Then, by the preceding Lemma,

$$|\Sigma_m \gamma_m f_k(z + c_m u) - f_k(z)| < \frac{1}{k} \tag{13}$$

for all complex u, and $z \in Z_0$.

The range H_0 of the function $h_0(u)$ on the complex numbers u is denumerable. Let H_0 be ordered as a sequence: $H_0=\{h_{01}, h_{02}, \ldots\}$, and put $h_{0l} = e^{a_{0l}+ib_{0l}}$ (a_{0l}, b_{0l} real). Select a sequence of positive integers n_k such that

$$\lim_{k\to\infty} e^{i(a_{0l}r_{n_k}+b_{0l}s_{n_k})} = \varepsilon_l$$

exists for $l = 1, 2, \ldots$

Now define $f(u)$ by

$$\begin{aligned} f(u) &= 0 && \text{if } u \in Z_0 \\ &= \varepsilon_l && \text{if } u \in \tilde{Z}_0,\ h_0(u) = h_{0l}. \end{aligned}$$

Then $f(u) = \lim_{k\to\infty} f_{n_k}(u)$ for all complex u. Obviously, $f(u) = 0$ on Z_0, $|f(u)| = 1$ on $\tilde{Z}_0$. Inequality (13) implies that $f(u)$ satisfies (2) for $z \in Z_0$ identically in u.

As in Theorem 1, the constructed solution is periodic. Every $z \in Z_0$ is a period of $f(u)$.

5. Let $c_m = \varepsilon_n^m$ where ε_n is a primitive n-th root of unity. Then equation (12) is satisfied with $r = 0$, $s = 1$, $\gamma_1 = \gamma_2 = \ldots = \gamma_n = 1/n$. If we let H be a HAMEL basis of the complex numbers over the field $C(\varepsilon_3, \varepsilon_4, \varepsilon_5, \ldots)$ then Z_0 as defined in 4. is independent of n and so is the function $f(u)$ (with $r = 0$, $s = 1$) as defined in (11). Hence, $f(u)$ satisfies, for $z \in Z_0$, simultaneously the equations

$$f(z) = \frac{1}{n} \sum_{m=1}^{n} f(z + \varepsilon_n^m u) \quad (n = 1, 2, \ldots) \tag{14}$$

and we have

Theorem 4. *There exists a bounded function $f(u)$, not equivalent to a constant, which at every z from a non-measurable everywhere dense set Z_0 of power c equals the arithmetic average of $f(u)$ over the vertices of any regular polygon with center at z.*

If $\gamma_1 = \gamma_2 = \ldots = \gamma_n = 1/n$ and the c_m are real positive numbers, $a_m = \log c_m$, then putting $\xi = \exp(ir)$ in equation (12) we have $\sum_{m=1}^{n} \xi^{a_m} = 0$ to be satisfied by some ξ of modulus 1. Clearly there exists a ξ satisfying these conditions if the a_m form an arithmetic progression, that is the c_m form a geometric progression.

Another noteworthy case is $\sum_{m=1}^{n} \gamma_m = 0$. Then equation (12) is satisfied by $r = s = 0$, no matter what the c_m are.

If the $c_1, c_2, \ldots, c_n$ are multiplicatively independent, that is

$$c_1^{p_1}, c_2^{p_2}, \ldots c_n^{p_n} = 1$$

for integers $p_1, p_2, \ldots, p_n$ implies $p_1 = p_2 = \ldots = p_n = 0$, then at least one of the n-tuples $a_1, \ldots, a_n; b_1, \ldots, b_n$ is rationally independent. In this case it is readily seen that condition (3) is equivalent to

$$\max |\gamma_m| \leq \tfrac{1}{2} \Sigma |\gamma_m|, \tag{15}$$

a condition in which the c_m do not enter at all.

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