GEOMETRICAL EXTREMA SUGGESTED BY A LEMMA OF BESICOVITCH

PAUL BATEMAN, The Institute for Advanced Study, and PAUL ERDŐS, Syracuse University

1. Introduction. In [1] Besicovitch needed as a lemma a result of the following type.

THEOREM 1. *Given a set Γ of coplanar circles, the center of no one of them being in the interior of another, and U the circle (or a circle) of Γ whose radius does not exceed the radius of any other circle of Γ, then the number of circles meeting U does not exceed* 18.

Besicovitch proved the weaker theorem obtained from this one by replacing 18 by 21. In this paper we shall prove Theorem 1 as it stands. The number 18 cannot be replaced by a smaller number, as is shown by the example in which all of the circles have radius 1 and the centers are at the following points in a polar coordinate system: the origin, the points $(1, h\cdot 60°)$ where $h=0, 1, \cdots, 5$, and the points $(2 \cos 15°, (2k+1)\cdot 15°)$ where $k=0, 1, \cdots, 11$.

We prove Theorem 1 by establishing its equivalence with Theorem 2 and then proving the latter.

THEOREM 2. *It is impossible to have* 20 *points in* a circle of radius* 2 *such that one of the points is at the center and all of the mutual distances are at least* 1.

Naturally one can ask the general question: For any positive integer n, what is the radius $r(n)$ of the smallest† circle containing n points one of which is at the center and all the mutual distances between which are at least 1? Thus Theorem 2 says that $r(20)>2$. A related question is the following: Of all sets of n points in the plane such that the mutual distances are all at least 1 (with no restriction on the arrangement of the points) what set has the minimum diameter‡ $D(n)$? The following theorem answers this question for $n=7$.

THEOREM 3. *A set of seven points in the plane whose mutual distances are all at least* 1 *has diameter at least* 2, *with this value being attained only by the set of points consisting of the vertices and circumcenter of a regular hexagon of side-length* 1.

The asymptotic behavior of $r(n)$ and $D(n)$ is well-known§; in fact, for large n the regular hexagonal lattice gives about the best results, so that $D(n)\sim 2r(n) \sim(12/\pi^2)^{1/4}n^{1/2}$ as $n\to\infty$. However we are interested here in small values of n.

* The term "in" is supposed to include the boundary.

† It is not difficult to see that the greatest lower bound is attained here.

‡ The diameter of a set of points is the least upper bound of all the mutual distances. For a finite set this is simply the greatest mutual distance. Also note that the diameter of a polygon is equal to the diameter of the point-set consisting of the vertices.

§ See [3], [4], [5], [6].

For each n the values of $r(n)$ and $D(n)$ can be calculated to any desired degree of accuracy by constructing a sufficiently fine mesh, but the proofs of our theorems will show that to accomplish this practically is not easy.

The first few values of $r(n)$ are as follows: $r(2)=r(3)=\cdots=r(7)=1$, $r(8)=\frac{1}{2}$ cosec $(180°/7)=1.15\cdots$, $r(9)=\frac{1}{2}$ cosec $22°.5=1.30\cdots$, $r(10)=\frac{1}{2}$ cosec $20°=1.46\cdots$, $r(11)=\frac{1}{2}$ cosec $18°=1.61\cdots$, $r(12)=1.68\cdots$, and $r(13)=\sqrt{3}=1.73\cdots$. The evaluation of $r(8)$ through $r(11)$ will come out in the proof of Theorem 2; the evaluation of $r(12)$ and $r(13)$ can be effected by similar methods. Further, the example before the statement of Theorem 2 shows that $r(19)\leqq 2\cos 15°=1.93\cdots$. Also $r(20)>2$.

The first few values of $D(n)$ are also easy to find: $D(2)=D(3)=1$, $D(4)=\sqrt{2}=1.41\cdots$, $D(5)=\frac{1}{2}(1+\sqrt{5})=1.61\cdots$, $D(6)=2\sin 72°=1.90\cdots$. Further, Theorem 3 gives $D(7)=2$. For $n=3, 4, 5$ the vertices of the regular n-gon of side-length 1 give the minimum diameter; for $n=6$ the best result is given by the set of points consisting of the vertices and circumcenter of a regular pentagon of circumradius 1.

2. Proof that Theorem 1 is equivalent to Theorem 2. To show that Theorem 1 implies Theorem 2 we proceed as follows. Suppose we have k points in a circle of radius 2 such that one of the points is at the center and all the mutual distances are at least 1. If we construct a circle of unit radius about each point and then apply Theorem 1 to this set of k circles, with U as the circle around the central point, we get $k\leqq 19$.

In showing that Theorem 2 implies Theorem 1 (and, in fact, in proving Theorem 2) we use a polar coordinate system with radial distance ρ and amplitude θ. The set of points such that $\rho\leqq t$ we denote by $C(t)$. Now we may assume that the U of Theorem 1 is the unit circle of our polar coordinate system. Thus we have a set Δ of $k-1$ circles of radius at least 1 each one of which meets the the unit circle U, the center of no one of these k circles (including U) lying in the interior of another. Then it suffices to show that we can construct a set Δ^* of $k-1$ points in $C(2)$ whose mutual distances and distances from the origin are all at least 1. We do this by choosing a point for Δ^* corresponding to each circle D of Δ in the following way: if the center of D lies in $C(2)$, we pick the center; if the center X of D lies outside of $C(2)$, we pick the point R lying on the circle $\rho=2$ and having the same amplitude as X. As a result of this correspondence the circle of radius 1 about a point Q of Δ^* is contained in the corresponding circle of Δ; hence Q is at distance at least 1 from the points of Δ^* which were originally centers of circles of Δ. Thus it remains only to prove that if two circles of Δ have centers X and Y both of which lie outside of $C(2)$, then the corresponding points R and S of Δ^* have mutual distance at least 1. Let $OX=x$, $OY=y$, angle $XOY=\psi$. Then by the properties of Δ we have $\overline{XY}^2\geqq\max\{(x-1)^2, (y-1)^2\}$; that is,

$$x^2+y^2-2xy\cos\psi\geqq\max\{(x-1)^2, (y-1)^2\}.$$

Now if we suppose that $y\geqq x$ we have

$$\cos\psi \leqq \frac{x^2+y^2-(y-1)^2}{2xy} = \frac{1}{x} + \frac{x^2-1}{2xy}$$
$$\leqq \frac{1}{x} + \frac{x^2-1}{2x^2} = 1 - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 \leqq \frac{7}{8},$$

since $x>2$. Therefore $\overline{RS}^2 = 8-8\cos\psi \geqq 1$. Thus the points chosen for Δ^* have the desired properties and our proof of equivalence is finished.

3. Proof of Theorem 2. The proof is based upon the following lemmas, of which the first is basic and the others are simple corollaries thereof.

LEMMA 1. *Let r and R be such that $0<R-1\leqq r\leqq R$ and suppose that we have two points P and Q which lie in the annulus $r\leqq\rho\leqq R$ and which have mutual distance at least* 1. *Then the minimum $\phi(r, R)$ of the angle POQ has the following values*

$$\phi(r, R) = \arccos\frac{R^2+r^2-1}{2Rr}, \qquad \text{if } R-1\leqq r\leqq R-\frac{1}{R};$$

$$\phi(r, R) = \arccos\left(1-\frac{1}{2R^2}\right) = 2\arcsin\frac{1}{2R}, \qquad \text{if } R-\frac{1}{R}\leqq r\leqq R.$$

In particular

$\phi(1, 1.10) > 54.^\circ 0$	$\phi(1.10, 2) > 16.^\circ 8$
$\phi(1, 1.15) > 51.^\circ 5$	$\phi(1.15, 2) > 20.^\circ 0$
$\phi(1, 1.20) > 49.^\circ 2$	$\phi(1.20, 2) > 22.^\circ 3$
$\phi(1, 1.25) > 47.^\circ 1$	$\phi(1.25, 2) > 24.^\circ 1$
$\phi(1, 1.30) > 45.^\circ 2$	$\phi(1.30, 2) > 25.^\circ 5$
$\phi(1, 1.45) > 40.^\circ 3$	$\phi(1.45, 2) > 28.^\circ 3$
$\phi(1, 1.60) > 36.^\circ 4$	$\phi(1.60, 2) > 28.^\circ 9$

Proof. It suffices to consider the case in which $OQ=R$ and $PQ=1$. If we put $OP=\rho$, then our problem is to find the minimum of $f(\rho)=$angle POQ $=\arccos\{(R^2+\rho^2-1)/(2R\rho)\}$ for ρ in the interval $r\leqq\rho\leqq R$. By differential calculus we see that although $f(\rho)$ may have an interior maximum for $\rho=(R^2-1)^{1/2}$, it cannot have an interior minimum. If we compare $f(r)=\arccos\{(R^2+r^2-1)/(2Rr)\}$ and $f(R)=\arccos\{1-1/(2R^2)\}$, we see that when $R-1\leqq r\leqq R-1/R$ the minimum is achieved for $OQ=R$, $PQ=1$, and $OP=r$, while when $R-1/R\leqq r\leqq R$ the minimum is achieved for $OQ=R$, $PQ=1$, and $OP=R$.

We shall find it convenient in what follows to use the term admissible points to refer to a set of points in $1\leqq\rho\leqq 2$ whose mutual distances are all at least 1.

LEMMA 2. *There are at most* 12 *admissible points in the annulus* $1.45 \leqq \rho \leqq 2$.

Proof. This follows from the fact that $13\phi(1.45, 2) > 360°$. (Note that the constant 1.45 could be replaced by any number r such that $\phi(r, 2) > 360°/13$, for example by 1.402.)

LEMMA 3. *It is impossible to have* 7 *admissible points in* $C(1.15)$, 8 *admissible points in* $C(1.30)$, 9 *admissible points in* $C(1.45)$, *or* 10 *admissible points in* $C(1.60)$.

Proof. In fact, $7\phi(1, 1.15) > 360°$, $8\phi(1, 1.30) > 360°$, $9\phi(1, 1.45) > 360°$, $10\phi(1, 1.60) > 360°$. (Note that the constants 1.15, 1.30, 1.45, 1.60 could be replaced by any values less than $\frac{1}{2}$ cosec $(180°/7)$, $\frac{1}{2}$ cosec $22°.5$, $\frac{1}{2}$ cosec $20°$, $\frac{1}{2}$ cosec $18°$, respectively. This remark, along with the fact that the vertices of the regular k-gon of side length 1 with center at the origin do constitute a set of k admissible points in $C\{\frac{1}{2}\text{ cosec }(180°/k)\}$, provides the evaluation of $r(8)$, $r(9)$, $r(10)$, $r(11)$).

LEMMA 4. *It is impossible to have* 7 *admissible points in* $C(1.30)$ *such that* 6 *of them lie in* $C(1.10)$.

Proof. If we had 7 admissible points in $C(1.30)$, 6 of which lay in $C(1.10)$, then of the 7 angles subtended at O by pairs of consecutive points (considered in order of amplitude) 5 would be each at least $\phi(1, 1.10)$ and the other two would be each at least $\phi(1, 1.30)$. But $5\phi(1, 1.10) + 2\phi(1, 1.30) > 360°.4$.

LEMMA 5. *It is impossible to have* 8 *admissible points in* $C(1.45)$ *such that* 7 *of them lie in* $C(1.25)$. *It is impossible to have* 8 *admissible points in* $C(1.45)$ *such that* 6 *of them lie in* $C(1.15)$.

Proof. The proof is similar to that of Lemma 4. The first part follows from the fact that $6\phi(1, 1.25) + 2\phi(1, 1.45) > 363°.2$; the second from the fact that $4\phi(1, 1.15) + 4\phi(1, 1.45) > 367°.2$.

Now we come to the proof of Theorem 2. We suppose that we have 19 admissible points and seek to get a contradiction. By Lemma 2 there are at most 12 admissible points outside $C(1.45)$ and by Lemma 3 at most 8 admissible points in $C(1.45)$. Thus there are two cases to consider, according to whether we have 7 admissible points in $C(1.45)$ and 12 admissible points outside, or 8 admissible points in $C(1.45)$ and 11 admissible points outside. The first case we subdivide further, according to whether one of the 7 points lies outside of $C(1.30)$ or not.

Case Ia: 12 admissible points $B_1, \cdots, B_{12}$ outside of $C(1.45)$, 7 admissible points $A_1, \cdots, A_7$ in $C(1.45)$, one of the A_i, say A_k, outside of $C(1.30)$. All the angles B_iOB_{i+1} exceed $\phi(1.45, 2) > 28°.3$. (We mean to include the angle $B_{12}OB_1$, of course; similarly in the sequel.) But for any i the angle A_kOB_i exceeds $\phi(1.30, 2) > 25°.5$ and so one of the angles B_iOB_{i+1} exceeds $2\phi(1.30, 2) > 51°.0$. However $11\ (28°.3) + 51°.0 = 362°.3$, a contradiction.

Case Ib: 12 admissible points $B_1, \cdots, B_{12}$ outside of $C(1.45)$, 7 admissible points $A_1, \cdots, A_7$ in $C(1.30)$. By Lemma 3 one of the A_i, say A_k, lies outside of $C(1.15)$ and by Lemma 4 another one of the A_i, say A_m, lies outside of $C(1.10)$. For each i the angle A_kOB_i exceeds $\phi(1.15, 2) > 20\overset{\circ}{.}0$ and the angle A_mOB_i exceeds $\phi(1.10, 2) > 16\overset{\circ}{.}8$. Hence an angular sector of more than $40\overset{\circ}{.}0$ and another sector of more than $33\overset{\circ}{.}6$ are ruled out as possible locations for points B_i. These sectors cannot overlap, since the angle A_kOA_m is at least $\phi(1, 1.30) > 45\overset{\circ}{.}2$. If one of the angles B_iOB_{i+1}, say B_nOB_{n+1}, includes both these sectors, then B_nOB_{n+1} exceeds $40\overset{\circ}{.}0 + 33\overset{\circ}{.}6 = 73\overset{\circ}{.}6$ and we have a contradiction, since $11\phi(1.45, 2) + 73\overset{\circ}{.}6 > 384\overset{\circ}{.}9$. Otherwise we see that out of the 12 angles B_iOB_{i+1} one exceeds $40\overset{\circ}{.}0$ and another exceeds $33\overset{\circ}{.}6$. Since at most 9 admissible points can lie in $C(1.60)$, at least 10 of the 12 points B_i lie outside $C(1.60)$. Hence at least 8 of the 12 angles B_iOB_{i+1} exceed $\phi(1.60, 2) > 28\overset{\circ}{.}9$. But $40\overset{\circ}{.}0 + 33\overset{\circ}{.}6 + 6(28\overset{\circ}{.}9) + 4\phi(1.45, 2) > 360\overset{\circ}{.}2$, a contradiction.

Case II: 11 admissible points $B_1, \cdots, B_{11}$ outside $C(1.45)$, 8 admissible point $A_1, \cdots, A_8$ in $C(1.45)$. By Lemma 3 one of the A_i, say A_k lies outside $C(1.30)$; by Lemma 5 a second one of the A_i, say A_m, lies outside of $C(1.25)$ and a third one of the A_i, say A_n, lies outside of $C(1.15)$.

Now for each i the angle B_iOA_k exceeds $\phi(1.30, 2) > 25\overset{\circ}{.}5$, the angle B_iOA_m exceeds $\phi(1.25, 2) > 24\overset{\circ}{.}1$, and the angle B_iOA_n exceeds $\phi(1.15, 2) > 20\overset{\circ}{.}0$. Hence three angular sectors of more than $51\overset{\circ}{.}0$. $48\overset{\circ}{.}2$, and $40\overset{\circ}{.}0$, respectively, are ruled out as possible locations for the points B_i. If one of the angles B_iOB_{i+1}, say B_pOB_{p+1}, includes two of these sectors, then B_pOB_{p+1} exceeds $20\overset{\circ}{.}0 + 24\overset{\circ}{.}1 + 40\overset{\circ}{.}3 = 84\overset{\circ}{.}4$, since A_kOA_m, A_mOA_n, and A_nOA_k are each at least $\phi(1, 1.45) > 40\overset{\circ}{.}3$; this gives a contradiction, since $10\phi(1.45, 2) + 84\overset{\circ}{.}4 > 367\overset{\circ}{.}4$.

On the other hand if no angle B_iOB_{i+1} includes two of the proscribed sectors, then we see that out of the 12 angles B_iOB_{i+1} one exceeds $51\overset{\circ}{.}0$, another exceeds $48\overset{\circ}{.}2$, another exceeds $40\overset{\circ}{.}0$, and each of the 8 remaining exceeds $\phi(1.45, 2) > 28\overset{\circ}{.}3$. But $51\overset{\circ}{.}0 + 48\overset{\circ}{.}2 + 40\overset{\circ}{.}0 + 8(28\overset{\circ}{.}3) = 365\overset{\circ}{.}6$, a contradiction.

4. Proof of Theorem 3. We require the following lemmas, of which the first three are well-known.

LEMMA 6. *The area of a triangle does not exceed $\frac{1}{4}\sqrt{3}$ times the square of the longest side.*

LEMMA 7. *The product of the diagonals of a quadrilateral is at least twice the area.*

LEMMA 8. *If a convex polygon has perimeter not less than 2π, its diameter exceeds 2.*

LEMMA 9. *If there is a point in a triangle whose distance from each vertex is at least 1, then some side of the triangle has length at least $\sqrt{3}$.*

Proof. One of the sides of the triangle must subtend an angle of $120°$ or more at the point in question.

LEMMA 10. *If $0<\xi\leqq\frac{1}{3}\pi$ and $ABCD$ is a convex quadrilateral with $\sphericalangle ABC=\frac{2}{3}\pi+\xi$, $\sphericalangle BCD\geqq\frac{2}{3}\pi-\frac{2}{3}\xi$, and $1\leqq AB$, BC, $CD\leqq 2$, then $AD>2$.*

Proof. Put $AB=x$, $BC=y$, $CD=z$, $\sphericalangle BCD=\theta$. Then

$$\begin{aligned}\overline{AD}^2 &= \left\{y - x\cos\left(\tfrac{2}{3}\pi+\xi\right) - z\cos\theta\right\}^2 + \left\{x\sin\left(\tfrac{2}{3}\pi+\xi\right) - z\sin\theta\right\}^2 \\ &= x^2+y^2+z^2-2yz\cos\theta+2zx\cos\left(\theta+\tfrac{2}{3}\pi+\xi\right)-2xy\cos\left(\tfrac{2}{3}\pi+\xi\right).\end{aligned}$$

Considering this expression as a function of x, y, z, θ over the domain $1\leqq x$, y, $z\leqq 2$, $\frac{2}{3}\pi-\frac{2}{3}\xi\leqq\theta\leqq\pi$, we see from the positivity of the partial derivatives that the smallest value of $\overline{AD}^2$ occurs for $x=y=z=1$, $\theta=\frac{2}{3}\pi-\frac{2}{3}\xi$. It suffices therefore to prove that

$$f(\xi) = 3 - 2\cos\left(\tfrac{2}{3}\pi-\tfrac{2}{3}\xi\right) + 2\cos\left(\tfrac{4}{3}\pi+\tfrac{1}{3}\xi\right) - 2\cos\left(\tfrac{2}{3}\pi+\xi\right)$$

exceeds 4 for $0<\xi\leqq\frac{1}{3}\pi$; but this follows from the fact that $f(0)=4$, $f(\frac{1}{3}\pi)=5-4\cos(4\pi/9)$, and $f''(\xi)<0$ for $0\leqq\xi\leqq\frac{1}{3}\pi$.

LEMMA 11. *If $ABCDE$ is a convex pentagon such that AB, BC, CD, $DE\geqq 1$, $AE\geqq\sqrt{3}$, and $\sphericalangle C>120°$, then either the diagonal $AD>2$ or the diagonal $BE>2$.*

Proof. If $\sphericalangle A>90°$, then $BE>2$; if $\sphericalangle E>90°$, then $AD>2$. Suppose then that neither angle adjacent to AE exceeds 90°. Then if $\sphericalangle C=120°+\xi$, we see that either $\sphericalangle B\geqq 120°-\frac{1}{2}\xi$ or $\sphericalangle D\geqq 120°-\frac{1}{2}\xi$, for otherwise the angle sum of the pentagon would be less than $90°+90°+(120°+\xi)+2(120°-\frac{1}{2}\xi)=540°$. Hence by Lemma 10 either $AD>2$ or $BE>2$.

Now we turn to the proof of Theorem 3. Suppose then that we have seven points in the plane such that the mutual distances are all at least 1. In proving that the diameter of the set of seven points is at least 2, we consider five cases, according as the convex hull of the seven points is a triangle, quadrilateral, pentagon, hexagon, or heptagon. As stated in the theorem we shall find that the diameter actually exceeds 2 throughout, except for the subcase of the hexagonal case in which the angles of the hexagon are all 120°, the sides all have length 1, and the seventh point is at the center.

The triangular case. Suppose that a circle of radius $\frac{1}{2}$ is drawn about each of the seven points as center. Then no two of these circles can properly intersect. If one side of the triangle is intersected by the circles around two of the four inner points, then that side has length at least $3\sqrt{3}/2$ by the Pythagorean theorem. If no side of the triangle is intersected by two of the circles about inner points, then the area of the triangle is greater than the area of three circles of radius $\frac{1}{2}$, namely $3\pi/4>\sqrt{3}$; thus by Lemma 6 at least one side of the triangle has length greater than 2.

The quadrilateral case. Again we construct a circle of radius $\frac{1}{2}$ about each of the seven points. If one side of the quadrilateral is intersected by the circles about two of the three inner points, then that side has length at least $3\sqrt{3}/2>2$. Suppose then that no side is intersected by more than one of the circles about the inner points. If just one or two sides are intersected by inner circles, then the area of the quadrilateral is greater than the area of three circles of radius $\frac{1}{2}$, namely $3\pi/4>2$, and thus by Lemma 7 at least one of the diagonals of the quadrilateral has length greater than 2.

Now notice that if a side of the quadrilateral is intersected by the circle around one of the inner points, then that side has length at least $\sqrt{3}$ by the Pythagorean theorem. If three sides are intersected by inner circles and the quadrilateral is a rectangle, then all sides of the rectangle have length at least $\sqrt{3}$ and hence the diagonals have length at least $\sqrt{6}$. If three sides are intersected by the circles about inner points and one of the angles of the quadrilateral exceeds 90°, then at least one of the sides adjoining that angle has length $\sqrt{3}$ or greater, while the other side adjoining it has length at least 1; hence (by the law of cosines) the diagonal spanning this angle has length greater than 2.

The pentagonal case. Again if the circles of radius $\frac{1}{2}$ about the two inner points intersect the same side of the pentagon, then that side has length at least $3\sqrt{3}/2$. On the other hand if these two circles intersect two different sides of the pentagon, then both these sides have lengths at least $\sqrt{3}$, the pentagon has perimeter at least $3+2\sqrt{3}>2\pi$, and thus by Lemma 8 the diameter of the pentagon exceeds 2.

Thus we assume that at least one of the inner points, say K, has a distance from the perimeter of the pentagon greater than $\frac{1}{2}$. Then the other inner point, say L, must lie in one of the triangles formed by K and consecutive vertices of the pentagon, say the triangle KAB. By Lemma 9 one of the sides of the triangle KAB has length at least $\sqrt{3}$. If either AK or BK has length $\sqrt{3}$ or more, the diameter of the pentagon exceeds $\sqrt{3}+\frac{1}{2}>2$. We suppose then that $AB \geqq \sqrt{3}$. Since one of the angles ALK and BLK is at least 90°, either $AK \geqq \sqrt{2}$ or $BK \geqq \sqrt{2}$. Suppose $AK \geqq \sqrt{2}$. Then prolong AK until it meets the pentagon again at a point X. If $AX>2$, we are finished. If $AX \leqq 2$, then $KX \leqq 2-\sqrt{2}$, the side on which X lies has length at least $2\{1-(2-\sqrt{2})^2\}^{1/2}>1.62$, the pentagon has perimeter greater than $3+1.73+1.62=6.35>2\pi$, and thus the pentagon has diameter more than 2 by Lemma 8.

The heptagonal case. This case is settled immediately by Lemma 8. Actually by pushing our methods further it is possible to prove that of all convex heptagons with sides of length at least 1 the minimum diameter occurs for the regular heptagon of side-length 1, in which case it is $1+2\cos(2\pi/7)=2.24\cdots$.

The hexagonal case. Dismissing the case in which a side of the hexagon has length greater than 2, we separate the proof into cases according to the number and arrangement of those angles of the hexagon which exceed 120°. First of all we note that if two adjacent angles of the hexagon exceed 120°, the diagonal spanning these two angles has length greater than 2. This case occurs, for example, if there are four or five angles of the hexagon exceeding 120°.

If exactly one angle of the hexagon exceeds 120°, say is $120°+\xi$, then one of the angles adjacent to it is at least $120°-\frac{1}{2}\xi$; the proof is completed in this case by Lemma 10. If exactly two angles of the hexagon exceed 120°, say have the values $120°+\alpha$ and $120°+\beta$, and these two angles are non-adjacent, then somewhere in the hexagon we have a pair of adjacent angles of which one is $120°+\xi$, $\xi>0$, and the other is at least $120°-\frac{2}{3}\xi$; for otherwise the pentagon would have an angle sum less than $(120°+\alpha)+(120°+\beta)+(120°-\frac{2}{3}\alpha)+(120°-\frac{2}{3}\beta)+(120°-\frac{2}{3}\max\{\alpha, \beta\})+120° \leqq 720°$. Again the proof is completed by Lemma 10.

In case that exactly three of the angles of the hexagon exceed 120° and no two of these are consecutive, let A, B, C, D, E, F be the vertices of the hexagon and let A, C, E be those at which the angles exceed 120°. First we remark that if one of the sides of the hexagon has length greater than $\frac{1}{2}(\sqrt{13}-1)$, we have a diagonal of length greater than 2 by the law of cosines; thus we may assume all sides of the hexagon to have length not exceeding $\frac{1}{2}(\sqrt{13}-1)<\sqrt{3}$. The three diagonals AC, CE, EA divide the hexagon up into four triangles. One of these four triangles contains the seventh point and hence one of the three diagonals AC, CE, EA has length at least $\sqrt{3}$ by Lemma 9. Suppose $EA \geqq \sqrt{3}$. Then by applying Lemma 11 to the pentagon $ABCDE$ we find a diagonal of length greater than 2.

The only case left is that in which no angle of the hexagon exceeds 120°, *i.e.*, all angles are 120°. We easily see that every diagonal spanning two angles has length at least 2 and that the diameter is exactly 2 if and only if all the sides of the hexagon have length 1 (with the seventh point at the center naturally).

5. Further questions. Another function which we could consider is the diameter $d(n)$ of the smallest circle containing n points whose mutual distances are all at least 1, without the restriction that one point be at the center. Obviously $D(n) \leqq d(n) \leqq 2r(n)$ and the same general remarks that were made about $D(n)$ and $r(n)$ could be made about $d(n)$. The first few values of $d(n)$ are $d(2)=1$, $d(3)=2/\sqrt{3}=1.15\cdots$, $d(4)=\sqrt{2}=1.41\cdots$, $d(5)=\operatorname{cosec} 36°=1.70\cdots$, $d(6)=2$, $d(7)=2$.

Naturally we can consider the analogue of $d(n)$ for other figures than the circle, for example, the side-length $t(n)$ of the smallest equilateral triangle containing n points whose mutual distances are all at least 1. An interesting unsolved question about $t(n)$ is whether or not $t(\frac{1}{2}k\{k+1\}+1)>k-1$ for k a positive integer. Obviously $t(\frac{1}{2}k\{k+1\}) \leqq k-1$, since the regular hexagonal lattice gives $\frac{1}{2}k\{k+1\}$ admissible points in a triangle of side-length $k-1$.

In the proof that $D(7)=2$ we remarked that of all convex heptagons whose sides have length 1 or more the minimum diameter is assumed by the regular heptagon of side-length 1. An analogous statement can be made for triangles, quadrilaterals, and pentagons. However for hexagons the minimum diameter is assumed by the equilateral hexagon of side-length 1 whose angles are alternately 90° and 150°. The situation for n-gons with $n>7$ is an open question.

Another interesting problem is whether or not $r(n+1)>r(n)$ for $n \geqq 7$ and whether $D(n+1)>D(n)$ for $n \geqq 3$. It follows from Theorem 3 that $D(8)>D(7)$

$=2$, since the diameter of a set of seven points (with mutual distances 1 or more) is greater than 2 except for one special configuration (for which we can make a separate argument).

Of course we can consider the analogues of $r(n)$ and $D(n)$ in k dimensions, say $r_k(n)$ and $D_k(n)$. Clearly $D_k(n)=1$ for $n\leqq k+1$. It would be interesting to have a good estimate for $D_k(k+2)$ from below. It is probably true that $D_k(k+2) \geqq \sqrt{4/3}$, but this seems difficult to prove. On the other side it is easy to give an example to show that $D_k(k+2)\leqq\sqrt{4k/(3k-2)}$. As for $r_k(n)$, we do not even know the smallest value of n such that $r_k(n)>1$, except for $k=2$. For $n\to\infty$ there are asymptotic relations of the form (cf. [2])

$$\tfrac{1}{2}D_k(n) \sim r_k(n) \sim c_k n^{1/k},$$

but c_k is not known for $k>2$.

6. Added in Proof: After this paper was submitted for publication, a proof of our Theorems 1 and 2 was published by E. F. Reifenberg, Math. Gaz., vol. 32, 1948, pp. 290–292. His proof is in general similar to ours, although considerably different in detail.

The problem mentioned briefly in the third paragraph of §5 has been discussed recently in a paper by S. Vincze, Acta Univ. Szeged. (Sect. Sci. Math.) vol. 12, part A, 1950, pp. 136–142. For n not a power of 2 he shows that the minimum diameter possible for convex n-gons whose sides have length at least 1 is $\frac{1}{2}\operatorname{cosec}(\pi/2n)$. In particular for even n having at least one odd prime factor the regular n-gon of side length 1 does not give the minimal diameter. Vincze points out that a closely related problem was considered earlier by K. Reinhardt, Jber. Deutsch. Math. Verein., vol. 31, 1922, pp. 251–270.

References

1. A. S. Besicovitch, A general form of the covering principle and relative differentiation of additive functions, Proc. Cambridge Philos. Soc., vol. 41, 1945, pp. 103–110.

2. R. A. Rankin, On the closest packing of spheres in n dimensions, Ann. of Math. (2), vol. 48, 1947, pp. 1062–1081.

3. B. Segre and K. Mahler, On the densest packing of circles, This MONTHLY, vol. 51, 1944, pp. 261–270.

4. A. Thue, Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene, Christiana Videnskabs-Selskabets Skrifter 1910, no. 1.

5. L. Fejes (Tóth), Extremal distributions of points in the plane, on the surface of the sphere and in space, Acta Sci. Math. Nat. (Univ. Francisco-Josephina, Kolozsvar) no. 23, 1944, iv+55 pp.

6. L. Fejes (Tóth), Über einem geometrischen Satz, Math. Zeit., vol. 46, 1940, pp. 83–85.