P. ERDÖS and J. F. KOKSMA: *On the uniform distribution modulo 1 of sequences* $(f(n, \theta))$.

(Communicated at the meeting of June 25, 1949.)

I. *Introduction.* In a former paper [1]) we treated lacunary sequences. Now, using an other method, we consider general sequences. For notation and definitions, see [1]). We prove

**Theorem 1.** *Let* $f(1, \theta), f(2, \theta), \ldots$ *be a sequence of real numbers, defined for each value of* $\theta$ *of the segment* $\alpha \leq \theta \leq \beta$, *such that* $f(n, \theta)$ *for* $n = 1, 2, \ldots$ *as a function of* $\theta$, *has a continuous derivative* $f'_\theta$ *and such that the expression*

$$f'_\theta(n_1, \theta) - f'_\theta(n_2, \theta)$$

*for each couple of positive integers* $n_1 \neq n_2$ *is either a non-decreasing or a non-increasing function of* $\theta$ *on* $\alpha \leq \theta \leq \beta$, *the absolute value of which is* $\geqq \delta$, *where* $\delta$ *denotes a positive number which does not depend on* $n_1$, $n_2$, *or* $\theta$. *Then for almost all* $\theta$ *the discrepancy* $D(N, \theta)$ *of the sequence satisfies the inequality*

$$ND(N, \theta) = O(N^{\frac{1}{2}} \log^{\frac{5}{2}+\varepsilon} N) \qquad (\varepsilon > 0). \quad . \quad . \quad . \quad . \quad (1)$$

Theorem 1 is a special case of the more general

**Theorem 2.** *Let* $f(n, \theta)$ *for* $n = 1, 2, \ldots$ *denote a real continuous function of* $\theta$ *on* $\alpha \leqq \theta \leqq \beta$ *and let*

$$\Phi(n_1, n_2, \theta) = f(n_1, \theta) - f(n_2, \theta) \text{ for } n_1 \neq n_2$$

*have a continuous derivative* $\Phi'_\theta$ *which is* $\neq 0$ *and either non-decreasing or non-increasing on* $\alpha \leqq \theta \leqq \beta$. *Put*

$$A(M, N) = \frac{1}{N^2} \sum_{n_1 = M+2}^{M+N} \sum_{n_2 = M+1}^{n_1 - 1} \operatorname{Max}\left(\frac{1}{|\Phi'_\theta(n_1, n_2, \alpha)|}, \frac{1}{|\Phi'_\theta(n_1, n_2, \beta)|}\right)$$

*and assume that for some constant* $\gamma \geqq 1$

$$NA(M, N) \leqq K_0 \log^\gamma N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2$$

*for all couples of positive integers* $M$, $N$ *where* $K_0$ *is a positive constant.*

*Then for almost all numbers* $\theta$ *in* $\alpha \leqq \theta \leqq \beta$ *the discrepancy* $D(N, \theta)$ *of the sequence* $f(1, \theta), f(2, \theta), \ldots$ *satisfies the inequality*

$$ND(N) = O(N^{\frac{1}{2}} \log^{\frac{\gamma + 4 + \varepsilon}{2}} N) \qquad (\varepsilon > 0).$$

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[1]) P. ERDÖS and J. F. KOKSMA, On the uniform distribution modulo 1 of lacunary sequences. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **52**, 264—273 (1949). (= Indag. Math. **11**, 79—88 (1949).)

**Remarks. 1.** It is clear that the functions $f(n, \theta)$ of Theorem 1 satisfy the assumptions of Theorem 2. For if one ranges the $N$ numbers

$$f(M+1, \theta),\ f(M+2, \theta), \ldots,\ f(M+N, \theta)$$

in order of magnitude, these numbers at each step increase with at least the amount $\delta$ and we find

$$\sum_{n_2=M+1}^{n_1-1} \frac{1}{|f'_\theta(n_1, \theta) - f'_\theta(n_2, \theta)|} < 2 \sum_{\mu=1}^{N} \frac{1}{\mu\delta} < \frac{2}{\delta} \log 3N,$$

hence

$$NA(M, N) \leqq \frac{2}{\delta} \log 3N = O(\log^\gamma N) \text{ for } \gamma = 1.$$

2. As Mr J. W. S. CASSELS has shown us, he also proved Theorem 1. His very interesting method is different from ours. The proofs are completely independent from each other.

II. *Some lemma's.*

**Lemma 1.** *Let $f(n, \theta)$ for $n = 1, 2, \ldots$ denote a real continuous function of $\theta$ on $\alpha \leqq \theta \leqq \beta$ and let*

$$\Phi(n_1, n_2, \theta) = f(n_1, \theta) - f(n_2, \theta) \text{ for } n_1 \neq n_2$$

*have a continuous derivative $\Phi'_\theta$ which is $\neq 0$ and either non-decreasing or non-increasing on $\alpha \leqq \theta \leqq \beta$. Finally put*

$$A_N = \frac{1}{N^2} \sum_{n_1=2}^{N} \sum_{n_2=1}^{n_1-1} \operatorname{Max}\left(\frac{1}{|\Phi'_\theta(n_1, n_2, \alpha)|}, \frac{1}{|\Phi'_\theta(n_1, n_2, \beta)|}\right).$$

*Then we have for $N \geqq 2$, $h > 0$ ($h$ not depending on $n$ and $\theta$)*

$$\int_\alpha^\beta \left| \sum_{n=1}^{N} e^{2\pi i h f(n, \theta)} \right|^2 d\theta \leqq (\beta - \alpha) N + \frac{A_N}{h} N^2.$$

The proof of this lemma has been given by KOKSMA [3]).

**Lemma 2.** *If $u_1, u_2, \ldots$ is a real sequence and if $D(N)$ denotes its discrepancy then for each integer $m \geqq 1$, we have*

$$ND(N) \leqq K\left(\frac{N}{m+1} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h u_n} \right| \right),$$

*where $K$ denotes a numerical constant.*

This lemma is an improvement proved by ERDÖS–TURÁN [4]) of the one-dimensional case of a theorem of VAN DER CORPUT–KOKSMA [5]).

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[2]) For litt. see [1]) and also [5]).

[3]) J. F. KOKSMA, Ein mengentheoretischer Satz über die Gleichverteilung modulo Eins. Comp. Math. 2, 250—258 (1935).

[4]) P. ERDÖS and P. TURÁN, On a problem in the theory of uniform distribution. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 51, 1146—1154, 1262—1269 (1948). (= Indag. Math. 10, 370—378, 406—413 (1948).)

[5]) See J. F. KOKSMA, Diophantische Approximationen, Erg. d. Math. IV, 4 (1936), Kap. IX.

**Lemma 3.** *If $f(n, \theta)$ denotes the function of Lemma 1, and if $D(N, \theta)$ denotes the discrepancy of the sequence $f(1, \theta)$, $f(2, \theta)$, ...., then*

$$\int_\alpha^\beta N^2 D^2(N, \theta)\, d\theta \leqq K_1 (N \log^2 N + A_N N^2 \log N).$$

*where $K_1$ depends on $\beta - \alpha$ only.*

*Proof.* Putting $m = [\sqrt{N}]$, we have by Lemma 2

$$N^2 D^2(N, \theta) \leqq K^2 \left( N + 2\sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right| \right) +$$

$$+ K^2 \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right| \cdot \left| \sum_{n=1}^{N} e^{2\pi i k f(n,\theta)} \right|.$$

Hence

$$\int_\alpha^\beta N^2 D^2(N, \theta)\, d\theta \leqq K^2 \left( N(\beta - \alpha) + 2\sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \frac{1}{h} \int_\alpha^\beta \left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right| d\theta \right) +$$

$$+ K^2 \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \int_\alpha^\beta \left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right| \left| \sum_{n=1}^{N} e^{2\pi i k f(n,\theta)} \right| d\theta$$

and by the CAUCHY–SCHWARZ inequality for integrals

$$\leqq K^2 \left( N(\beta - \alpha) + 2\sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \frac{1}{h} \left\{ \int_\alpha^\beta 1^2\, d\theta \cdot \int_\alpha^\beta \left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right|^2 d\theta \right\}^{\frac{1}{2}} \right) +$$

$$+ K^2 \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \left\{ \int_\alpha^\beta \left| \sum_{n=1}^{N} e^{2\pi i h f(n,\theta)} \right|^2 d\theta \cdot \int_\alpha^\beta \left| \sum_{n=1}^{N} e^{2\pi i k f(n,\theta)} \right|^2 d\theta \right\}^{\frac{1}{2}}$$

$$\leqq K^2 \left( N(\beta - \alpha) + 2\sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \frac{1}{h} \left\{ (\beta - \alpha)^2 N + \frac{\beta - \alpha}{h} A_N \cdot N^2 \right\}^{\frac{1}{2}} \right) +$$

$$+ \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \left\{ (\beta - \alpha) N + \frac{1}{h} A_N \cdot N^2 \right\}^{\frac{1}{2}} \left\{ (\beta - \alpha) N + \frac{1}{k} A_N \cdot N^2 \right\}^{\frac{1}{2}}$$

by Lemma 1. Hence by the CAUCHY–SCHWARZ-inequality for sums

$$\int_\alpha^\beta N^2 D^2(N, \theta)\, d\theta \leqq K^2 \left( N(\beta - \alpha) + 2\sqrt{N} \left\{ \sum_{h=1}^{N} \frac{\beta - \alpha}{h} \sqrt{N} + \sum_{h=1}^{N} \frac{\sqrt{\beta - \alpha}}{h\sqrt{h}} \sqrt{A_N \cdot N} \right\} + \right.$$

$$\left. + \left\{ \sum_{h=1}^{N} \sum_{k=1}^{N} \frac{1}{hk} \left\{ (\beta - \alpha) N + \frac{1}{h} A_N \cdot N^2 \right\} \cdot \sum_{h=1}^{N} \sum_{k=1}^{N} \frac{1}{hk} \left\{ (\beta - \alpha) N + \frac{1}{k} A_N \cdot N^2 \right\} \right\}^{\frac{1}{2}} \right)$$

$$\leqq K_2 (N + N \log N + \sqrt{A_N}\, N^{\frac{3}{2}} + N \log^2 N + A_N N^2 \log N).$$

where $K_2$ only depends on $K$ and $\beta - \alpha$.

Now if

$$\sqrt{A_N}\, N^{\frac{3}{2}} > A_N N^2 \log N,$$

we should have

$$1 > \sqrt{A_N} \sqrt{N} \log N.$$

hence

$$A_N N^2 \log N < \sqrt{A_N}\, N^{\frac{3}{2}} < N.$$

Therefore

$$\int_{\alpha}^{\beta} N^2 D^2 (N, \theta)\, d\theta \leqq K_1 \left(N \log^2 N + A_N N^2 \log N\right).$$

Q.e.d.

**Lemma 4.** *Let $F(M, N) = F(M, N, \theta)$ denote a function of $\theta$ on a segment $\alpha \leqq \theta \leqq \beta$ for each couple of positive integers $M$ and $N$, such that*

$$|F(M, N)| \leqq |F(M, N_1)| + |F(M + N_1, N - N_1)| \quad . \ . \ . \quad (3)$$

*for each triple $M$, $N$ and $N_1 \leqq N$ and such that $F$ belongs to the class $L^2$ over the segment. Let further*

$$\int_{\alpha}^{\beta} |F(M, N, \theta)|^2\, d\theta \leqq K_3 N \log^{\sigma} N$$

*$K_3 > 0$ and $\sigma$, being real constants. Then for almost all $\theta$ in $\alpha \leqq \theta \leqq \beta$ we have*

$$|F(0, N, \theta)| = O\left(N^{\frac{1}{2}} \log^{\frac{\sigma+3+\varepsilon}{2}} N\right) \quad (\varepsilon > 0).$$

This lemma is a special case of a theorem of GÁL–KOKSMA, the proof of which will appear before long [6]).

III. We now prove Theorem 2.

Let $M$ denote an arbitrary integer $\geqq 1$ and consider the functions

$$f(M+1, \theta),\ f(M+2, \theta), \ldots; \quad . \ . \ . \ . \ . \ . \ . \quad (4)$$

these functions satisfy the assumptions of Lemma 1 and the corresponding number $A_N$ is exactly identical with the number $A(M, N)$ which we have defined in Theorem 2. Denoting the discrepancy of the sequence (4) by $D(M, N, \theta)$, we have by Lemma 3, applied to the sequence (4),

$$\int_{\alpha}^{\beta} N^2 D^2 (M, N, \theta)\, d\theta \leqq K_1 \left(N \log^2 N + A(M, N) N^2 \log N\right) \leqq K_4 N \log^{1+\gamma} N$$

because of (2). Now it is easily seen from the definition of $D(N)$, that if we put

$$F(M, N, \theta) = N D(M, N, \theta),$$

the relation (3) is satisfied. Hence Theorem 2 follows immediately from Lemma 4 with $\sigma = 1 + \gamma$.

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[6]) Cf. I. S. GÁL et J. F. KOKSMA, Sur l'ordre de grandeur des fonctions sommables. C. R. Acad. d. Sc. Paris, *227*, 1321—1323 (1948).