ON THE NUMBER OF TERMS OF THE SQUARE OF A POLYNOMIAL

BY

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Let $f_k(x) = a_0 + a_1 x^{n_1} + \dots + a_{k-1} x^{n_{k-1}}$, $a_i \neq 0$, for $0 \leqslant i \leqslant k-1$, a_i real, be a polynominal of k terms. Denote by $Q(f_k(x))$ the number of terms of $f_k(x)^2$. Put

$$Q(k) = \min Q(f_k(x)),$$

where $f_k(x)$ runs through all polynominals having k non-vanishing terms and real coefficients.

RÉDEI[1]) raised the problem whether $Q(k) < k$ is possible. RÉNYI, KALMÁR and RÉDEI[1]) proved in fact that $\liminf_{k\to\infty} Q(k)/k = 0$, also that $Q(29) \leqslant 28$. RÉNYI[1]) further proved that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{Q(k)}{k} = 0.$$

He also conjectured that

$$\lim_{k\to\infty} \frac{Q(k)}{k} = 0. \tag{1}$$

In this note we are going to prove (1), by a slight modification of the method used by RÉNYI. In fact we shall prove the following

Theorem. *There exist constants* $0 < c_2$ *and* $0 < c_1 < 1$, *so that*

$$Q(k) < c_2\, k^{1-c_1}. \tag{2}$$

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[1]) A. RÉNYI, Hungarica Acta Math. 1, p. 30—34 (1947).

First we state two lemmas, both of which are contained in RÉNYI's paper.

Lemma I. $Q(29) \leqslant 28$.

Lemma II. $Q(a.b) \leqslant Q(a).Q(b)$. (Lemma II in fact is almost obvious).

From lemmas I and II we immediately obtain that

$$Q(29^l) \leqslant 28^l, \tag{3}$$

or (2) is satisfied for integers of the form 29^l. Assume now $l \geqslant 2$ and $29^l < k < 29^{l+1}$.

Put

$$t = \left[\frac{l}{2}\right], \; r + 1 = 29^t. \tag{4}$$

Let $h(x) = a_0 + a_1 x^{n_1} + \ldots + a_r x^{n_r}$, $a_i \neq 0$, be the polynomial for which $h(x)^2$ has $Q(29^t) \leqslant 28^t$ terms. Consider now

$$F(x) = h(x)g(x), \, g(x) = b_0 + b_1 x^{n_r} + b_2 x^{2n_r} + \ldots + b_s x^{sn_r}, \; b_i \neq 0$$

where the b's and s will be determined later. Let us compute the number of terms of $F(x)$. Clearly $F(x)$ has exactly $(r-1)$ $(s+1)$ terms $d_u x^u$ where $u \not\equiv 0 \pmod{n_r}$, further the constant term and the coefficient of $x^{(s+1)n_r}$ can not be 0. By suitable choice of the b's we can clearly arbitrarily prescribe whether the coefficient of x^{vn_r}, $1 \leqslant v \leqslant s$ is 0 or not (we only have to solve equations of the first degree). Thus $g(x)$ can be so chosen that $F(x)$ should have $2 + (r-1)(s+1) + A$ terms where $0 \leqslant A \leqslant s$ is arbitrary. Put

$$s + 1 = \left[\frac{k-2}{r-1}\right]. \tag{5}$$

Clearly by (4) $s \geqslant r - 1$. Thus

$$2 + (r-1)(s+1) \leqslant k \leqslant 2 + (r-1)(s+1) + s.$$

Thus by what has been said before we can determine $g(x)$ so that $F(x) = g(x)h(x)$ has k terms. But then by lemma II $F(x)^2$ has not more than

$$(2s+2) . 28^t < c_2 \, k^{1-c_1} \quad \text{(by (4) and (5))}$$

terms $(g(x))^2$ has $\leqslant 2s+2$ terms). Thus our Theorem is proved.

It would be interesting to determine the order of $Q(k)$ more accurately. RÉNYI [2]) conjectured that $\lim_{k\to\infty} Q(k) = \infty$, but neither of us could prove this as yet.

One final remark: Since RÉNYI proves that $Q(29) \leq 28$ for polynominals having rational coefficients, our proof gives $Q(k) \leq c_2 k^{1-c_1}$ for polynominals with rational coefficients. RÉNYI [1]) asks whether $Q(k)$ is the same if the coefficients are rational, real, or complex.

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[2]) Oral communication.

SOME THEOREMS ON THE ROOTS OF POLYNOMIALS.

BY

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Our first theorem concerns a composition theorem for polynomials whose roots lie in certain sectors. It is an extension of a result of L. WEISSNER [1]) and can be proved by the same method. Here we state it, and we give a short independent proof, since it will be applied below.

The term *sector* is used here in the sense of an open point set in the complex plane bounded by two [2]) half lines starting from the origin. If S_α and S_β are sectors with aperture α and β, respectively, and if $\alpha + \beta \leq 2\pi$, then the product $S_\alpha S_\beta$, consisting of all points $w_1 w_2$ ($w_1 \varepsilon S_\alpha$, $w_2 \varepsilon S_\beta$) is also a sector, with aperture $\alpha + \beta$. The sector consisting of all points $-w$ ($w \varepsilon S$) is denoted by $-S$.

Theorem 1. *Put*

$$A(z) = \sum_0^M a_n z^n \qquad (a_M \neq 0)$$

$$B(z) = \sum_0^N b_n z^n \qquad (b_N \neq 0)$$

$$g(z) = \sum_0^K n!\, a_n b_n z^n \qquad (K = \mathrm{Min}\,(M, N))$$

Suppose that the roots of $A(z)$ all lie in the sector S_α ($\alpha \leq \pi$) and those of $B(z)$ in S_β ($\beta \leq \pi$). Then the roots of $g(z)$ all lie in the sector $S = -S_\alpha S_\beta$.

1) L. WEISSNER, Polynomials whose roots lie in a sector. Am. Journ. Math. **64**, 55—60 (1942).

2) A single half line in case of aperture 2π.