THE $\alpha+\beta$ HYPOTHESIS AND RELATED PROBLEMS

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1. Introduction. Let $a_1 < a_2 < a_3 \cdots$ be a set, A, of positive integers. Let $A(n)$ denote the number of integers of A which are not greater than n. The *asymptotic density* $\delta(A)$ of A is defined to be the lower limit of $A(n)/n$. This is a natural definition of density: the set of all (positive) integers has asymptotic density 1; the odd integers, $\frac{1}{2}$; the even integers, $\frac{1}{2}$; the integers which are multiples of 7, $1/7$.

A less natural definition, but one which has attracted considerable attention, is the *greatest lower bound* of $A(n)/n$. This is the *Schnirelmann density* [14] of A, denoted here by $d(A)$. The odd integers have density $\frac{1}{2}$, clearly, but the even integers have density 0 under this definition. In fact, if a set A lacks the integer 1, then $d(A)=0$. Although this may seem artificial, Schnirelmann was led to the definition quite naturally by a study of certain problems in additive number theory, as we shall see later.

It is clear from the definitions that $0 \leqq d(A) \leqq \delta(A) \leqq 1$.

Let B be the set of positive integers $b_1 < b_2 < b_3 < \cdots$. The sum $A+B$ of the sets A and B is defined as the set of integers of the form a_i or b_j or a_i+b_j; that is, an integer x is in $A+B$ if x is in A, or if x is in B, or if x is the sum of an integer of A and an integer of B. For convenience we denote $d(A)$, $d(B)$, and $d(A+B)$ by α, β, and γ, respectively. The question of the relation of γ to α and β has been the subject of much investigation, culminating in Mann's proof [12] of the celebrated Khintchine conjecture [8], commonly known as the $\alpha+\beta$ hypothesis, which is that γ is not less than the minimum of 1 and $\alpha+\beta$. This paper outlines briefly, and not at all exhaustively, the history and present state of knowledge of this problem and some related ones. Some of the simplest proofs are given.

Landau [11] and Rohrbach [13] have summarized much of the principal work in this field up to 1937 and 1938 respectively.

2. Asymptotic density. First we mention some well known sequences of integers and their asymptotic densities. If A is the set of prime numbers, then $\delta(A)=0$ [9]. The set of square-free integers has density $6/\pi^2$ [7]. If A is either the set of squares or the integers which are expressible as sums of two squares, then $\delta(A)=0$. The set of integers which can be written as sums of three or fewer squares has density $5/6$, as can be readily deduced from the fact that this set contains all integers except those of the form $4^a(7+8b)$.

Davenport [4] has shown that if A represents the set of integers which are sums of s or fewer cubes of positive integers, then $\delta(A)=1$ for $s=4$ but for no smaller value of s. Change cubes to fourth powers, change $s=4$ to $s=15$, and you get another result of Davenport [5]. In each of these cases the fact that the result is best possible can be proved by simple congruences. For example, no integer of the form $15+16a$ can be a sum of 14 fourth powers, because any fourth power is congruent to 0 or 1 (mod 16).

Less is known about the higher powers. If $A_{k,s}$ represents the set of integers which are expressible as sums of s or fewer kth powers of positive integers, it is known that $\delta(A_{k,k-1})=0$; but whether $\delta(A_{k,k})$ is zero or positive is not known for $k\geqq 3$.

We mentioned in §1 the problem of the relationship between the Schnirelmann densities of A, B, and $A+B$. For asymptotic densities the $\alpha+\beta$ hypothesis is not true: consider the case, for example, where both A and B are composed of all positive even integers, so that $A+B$ is the same set, and we have $\delta(A)=\delta(B)=\delta(A+B)=\frac{1}{2}$.

However, Erdös [6] has proved the following result. Let α', β', and γ' stand for $\delta(A)$, $\delta(B)$, and $\delta(A+B)$. If $\gamma'\leqq 1$, $\alpha'\leqq\beta'$, and $a_1=1$ (recall that a_1 is the first integer of A), then $\gamma'\geqq\frac{1}{2}\alpha'+\beta'$. This result is best possible in the sense that sets A and B can be exhibited so that the equality sign holds in the conclusion. This can be done by taking A and B to be the same set, the set of integers which are congruent to 0 or 1 (mod 4), so that $A+B$ is composed of all integers congruent to 0, 1, or 2 (mod 4).

3. Two proofs. Henceforth we shall discuss only Schnirelmann density. Schnirelmann [14] and Landau [10] proved that $\gamma\geqq\alpha+\beta-\alpha\beta$. We now prove this result.

Proof. Consider any integer a in A such that $a+1$ is not in A. Let the next largest integer in A be $a+h+1$. Thus we have a gap of length h, that is, a gap of exactly h integers which are not in A. Consider now any integer x, not necessarily in A. Let there be m gaps of lengths $h_1, h_2, \cdots, h_m$ in A among the integers which are $\leqq x$. In case x is not in A, we shall take h_m to be the number of consecutive integers $x, x-1, x-2, \cdots$ which are not in A; thus h_m, unlike the other h's, may not represent the length of a *complete* gap in the integers of A.

Now since $A(x)$ denotes the number of integers in A which are $\leqq x$, we have

$$x - A(x) = h_1 + h_2 + \cdots + h_m. \tag{1}$$

Moreover, a and $a+h+1$ being in A (but no integer between these two), we can add the $B(h)$ integers of B which are $\leqq h$ to a to get $B(h)$ integers in $A+B$ between a and $a+h+1$. Doing this for all the gaps of lengths $h_1, h_2, \cdots, h_m$ we see that, denoting $A+B$ by C,

$$C(x) \geqq A(x) + B(h_1) + B(h_2) + \cdots + B(h_m). \tag{2}$$

But by definition $B(h)\geqq\beta h$, and hence (2) implies

$$C(x) \geqq A(x) + \beta(h_1 + h_2 + \cdots + h_m),$$

which combines with (1) to give

$$C(x) \geqq A(x) + \beta(x - A(x)) = (1-\beta)A(x) - \beta x.$$

But $A(x) \geqq \alpha x$, so we have

$$C(x) \geqq (1 - \beta)\alpha x + \beta x = x(\alpha + \beta - \alpha\beta),$$

which completes the proof.

As we have stated the $\alpha+\beta$ hypothesis it includes the result that if $\alpha+\beta \geqq 1$, then $\gamma = 1$. This is very simple, and we prove it now.

Proof. We must prove that $A+B$ is the set of all integers. Suppose a certain integer n is not in $A+B$. Let $a_1 < a_2 < a_3 < \cdots < a_k$ be the members of A which are $< n$. Then $A(n) = k$ and $k/n = A(n)/n \geqq \alpha$. Also it is clear that B lacks the $k+1$ integers $n-a_1, n-a_2, \cdots, n-a_k, n$. Hence β, the density of B, is at most $1-(k+1)/n$. Thus we have

$$\alpha \leqq \frac{k}{n}, \qquad \beta \leqq 1 - \frac{k+1}{n}, \qquad \alpha + \beta \leqq 1 - \frac{1}{n},$$

which contradicts the hypothesis.

4. Concerning Goldbach's hypothesis. Schnirelmann [14] proved that if A represents the set of all primes (we include 1 as a prime for convenience in this discussion), then, although $d(A) = 0$, $d(A+A)$ is positive. By the Schnirelmann-Landau theorem of §3 (or a fortiori by the $\alpha+\beta$ hypothesis) it follows from $d(A+A) = d(2A) = \lambda > 0$ that $d(4A) \geqq \lambda + \lambda - \lambda^2$ which exceeds $\frac{3}{2}\lambda$ if $\lambda < \frac{1}{2}$. By induction we know that there exists an n such that $d(nA) > \frac{1}{2}$ and the second result proved in §3 implies that $d(2nA) = 1$. Thus Schnirelmann was able to show the existence of a constant $2n$ such that every integer is expressible as a sum of $2n$ primes. Later, Vinogradoff proved by different methods that every sufficiently large odd integer is a sum of three primes. Goldbach's hypothesis is that every even integer is a sum of two primes.

5. The $\alpha+\beta$ hypothesis. Khintchine [8] conjectured the $\alpha+\beta$ hypothesis (that $\gamma \geqq \alpha+\beta$ or $\gamma = 1$, a best possible result), having proved it in the special cases $\alpha = \beta$ and $\alpha = 1 - 2\beta$. There followed a series of papers proving partial or modified results.

The Schnirelmann-Landau result has already been mentioned in §3. Besicovitch [2] defined β^* as the greatest lower bound of $B(n)/(n+1)$, and proved that the density of integers of the form a_i or $a_i + b_j$ is not less than $\alpha + \beta^*$, a result which is best possible.

Schur [15] proved that $\gamma \geqq \alpha/(1-\beta)$ or $\gamma = 1$. Brauer [3] proved that $\gamma \geqq \frac{9}{10}(\alpha+\beta)$ or $\gamma = 1$.

Landau [11, p. 7] posed the question of the as yet undecided problem and wrote: "Ich weiss es nicht: dies ungelöste Problem möchte ich dem Leser ans Herz legen."

Finally Mann [12] became interested in the problem while in attendance at a series of lectures on number theory by A. T. Brauer, and achieved the result that had eluded so many, the proof of the $\alpha+\beta$ hypothesis. Later Artin and Scherk [1] gave a simplification of Mann's proof.

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