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Ramanujan sums and almost periodic functions

by

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Introduction. Several classical formal trigonometrical expansions of the analytic theory of numbers have recently been shown [1]) to be periodic or almost periodic Fourier series of the functions which they represent. The object of the present paper is to prove a corresponding result for an extensive class of multiplicative arithmetical sequences.

In particular, it will be shown that the celebrated formal *trigonometrical series* of Ramanujan [2]) are almost periodic *Fourier series* in the sense of Besicovitch [3]). Hence the Ramanujan coefficients will turn out to be Fourier averages which vanish for incommensurable values of the frequency parameter, the almost periodic functions in question being always limit-periodic. It should be emphasized that the fact that Ramanujan's trigonometrical expansions turn out to be Fourier expansions leads without any further device to his explicit formulae, if one writes down the Fourier average representation of the coefficients.

Although the arithmetical functions $f(n)$ will only be considered for $n=1,2,\dots$, one can realize the usual assumption of the Besicovitch theory by placing $f(-n)=f(n)$ for $n=1,2,\dots$

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[1]) A. Wintner, Amer. Jour. Math. 57 (1935) p. 534—538; Duke Math. Jour. 2 (1936) p. 443—446; Amer. Jour. Math. 59 (1937) p. 629—634; P. Hartman and A. Wintner, Travaux Inst. Math. Tbilissi 3 (1938) p. 113—119; P. Hartman, Amer. Jour. Math. 61 (1938) p. 66—74.

[2]) S. Ramanujan, Collected Papers, Cambridge, 1927, p. 179—199.

[3]) A. S. Besicovitch, Almost Periodic Functions, Cambridge, 1932, p. 91—112.

and $f(0)=0$ (the multiplicative character of f then remains preserved). It is understood that a class (B^l) of functions $f(n)$ which are defined for integers may be introduced either directly [4] or by considering the step function $f(t)$ which has the value $f(n)$ for $n \leqslant t < n+1$.

1. By a multiplicative function f is meant a sequence $f(n)$ $n=1,2,3,\ldots$ for which $f(n_1 n_2)=f(n_1)f(n_2)$ whenever $(n_1, n_2)=1$, and $f(n) \neq 0$ for at least one n (so that $f(1)=1$). In order to simplify the formulae, only those multiplicative $f(n)$ will be considered for which

$$(1) \quad f(n)=\prod_{p|n} f(p), \text{ i. e. } f(p)=f(p^2)=f(p^3)=\ldots, \quad (f(1)=1),$$

where the p denote prime numbers. An $f(n)$ which satisfies (1) will be called strongly multiplicative. A classical instance of (1) is

$$(2) \qquad f(n)=\frac{\Phi(n)}{n}=\prod_{p|n}\frac{p-1}{p}=\prod_{p|n}\frac{\Phi(p)}{p};$$

(Φ = Euler's function).

For any $f(n)$ and for any positive integer k, put

$$(3) \qquad f^{(k)}(n)=1 \text{ or } f^{(k)}(n)=f(p_k) \text{ according as } n \not\equiv 0 \text{ or } n \equiv 0 \pmod{p_k},$$

where p_k is the k-th prime; and put

$$(4) \quad f_k(n)=\prod_{j=1}^{k} f^{(j)}(n), \text{ so that } f_k(n)=\prod_{n|p}{}' f(p), \text{ where } p \leqslant p_k.$$

According to (3), the function $f^{(k)}(n)$ of n has the period p_k and possesses the Fourier expansion

$$(5) \qquad f^{(k)}(n)=1+\frac{f(p_k)-1}{p_k}\sum_{m=0}^{p_k-1}\exp\left(2\pi i\frac{m}{p_k}n\right),$$

which is, in fact, nothing but the formula of equidistant trigonometrical interpolation. According to (4), the function $f_k(n)$ of n

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[4]) Cf. I. Seynsche, Rendic. Circ. Math. Palermo 55 (1931) p. 395—421, where Bohr's uniformly almost periodic case is considered.

has the period $P_k = p_1 p_2 \ldots p_{k-1} p_k$ and possesses, in view of (4) and (5), the Fourier expansion

$$(6)\quad f_k(n) = c_k + c_k \sum_{\substack{q | P_k \\ q>1}} \sum_{\substack{(m,q)=1 \\ 1 \leq m < q}} \prod_{p|q} \frac{f(p)-1}{f(p)-1+p} \cos\left(2\pi \frac{m}{q} n\right),$$

$$\text{where } c_k = \prod_{p \leq p_k} \left(1 + \frac{f(p)-1}{p}\right).$$

It is understood that the denominators in the product occurring in (6) are compensated by the factors of c_k.

2. For a function $g = g(n)$ defined for $n = 1, 2, 3, \ldots$, put

$$(7)\qquad M\{g\} = M\{g(n)\} = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} g(m),$$

if this mean value exists.

If $f(n)$ is strongly multiplicative and

$$(8)\qquad \sum \frac{|1-f(p)|}{p} < \infty,$$

then

$$(9)\qquad M\{f(n)\} = \prod \left(1 - \frac{1-f(p)}{p}\right).$$

In fact, it follows from the Möbius inversion formula that

$$f(1) + \ldots + f(n) = \Sigma \mu(k) F(k) \left[\frac{n}{k}\right], \text{ where } F(k) = \prod_{p|k} (1-f(p)).$$

On the other hand, it is clear from (8) and from Euler's factorization that

$$\sum \frac{|\mu(k) F(k)|}{k} = \prod \left(1 + \frac{1-f(p)}{p}\right).$$

Hence, $M\{f(n)\}$ exists and is represented by

$$M\{f(n)\} = \sum \frac{\mu(k) F(k)}{k} = \prod \left(1 - \frac{1-f(p)}{p}\right).$$

3. A corollary of (8) and (9) is that for a strongly multiplicative $f(n) \neq 0$ one has

$$(10)\qquad \lim_{n\to\infty}\frac{1}{\log n}\sum_{m=1}^{n}\frac{1}{mf(m)}=\prod\left(1-\frac{1-f(p)}{pf(p)}\right),\quad \text{if } \sum\frac{|1-f(p)|}{p\,|f(p)|}<\infty .$$

In fact, on writing $\frac{1}{f(p)}$ for $f(p)$ in (8) and (9), one obtains (10), since

$$(10\text{ bis})\quad \text{if } \frac{1}{n}\sum_{m=1}^{n}a_m\to\alpha,\ \text{then also } \frac{1}{\log n}\sum_{m=1}^{n}\frac{a_m}{m}\to\alpha .$$

Similarly, if $f(n)^l$ denotes the l-th power of $f(n)$, then

$$(11)\quad M\{f(n)^l\}=\prod\left(1-\frac{1-f(p)^l}{p}\right),\ \text{if } \sum\frac{|1-f(p)^l|}{p}<\infty,$$

where l is any real number. In fact, (11) follows from (8) and (9) by writing $f(n)^l$ for $f(n)$. Since

$$(11\text{ bis})\quad \text{if } \frac{1}{n}\sum_{m=1}^{n}a_m\to\alpha,\ \text{then also } \frac{1+\lambda}{n^{1+\lambda}}\sum_{m=1}^{n}m^{\lambda}a_m\to\alpha \text{ for every } \lambda>-1,$$

it follows from (11) that

$$\lim_{n\to\infty}\frac{1}{n^{1+\lambda}}\sum_{m=1}^{n}m^{\lambda}f(m)^{\lambda}=\frac{1}{\lambda+1}\prod\left(1-\frac{1-f(p)^{\lambda}}{p}\right),\quad \text{if } \sum\frac{|1-f(p)^{\lambda}|}{p}<\infty,\ \lambda>-1 .$$

Clearly, (10) may be interpreted as the limiting case $\lambda=-1$. Needless to say, the relation belonging to any $\lambda\geqslant -1$ is an essentially weaker statement than is (9) itself. In fact, the converse of (10 bis) or of (11 bis) is only true on Tauberian assumptions.

As an illustration, consider the example (2); so that $f(p)=1-p^{-1}$. Thus, (10) is applicable and goes over into Landau's relation

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{m=1}^{n} \frac{1}{\Phi(m)} = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)},$$

$$\text{since } \Pi\left(1+\frac{1}{p(p-1)}\right) = \frac{\Pi(1-p^{-6})}{\Pi(1-p^{-2})\,\Pi(1-p^{-3})};$$

while (11) is applicable for $-\infty < l < +\infty$ and gives SCHUR's relation

$$M\left\{\left(\frac{\Phi(n)}{n}\right)^l\right\} = \Pi\left(1-\frac{1}{p}\left(1-\frac{1}{p}\right)^l\right)$$

for every real l (and, as seen from the proof of (9) or (11), for every complex l also [5])).

4. For every strongly multiplicative, positive $f(n)$, let $f^+(n)$, $f^-(n)$ denote the strongly multiplicative, positive functions which at an arbitrary $n=p$ attain the values $f^+(p) = \max(1, f(p))$, $f^-(p) = \min(1, f(p))$, respectively. Then (2) shows that

(12_1) $f(n) = f^+(n)\, f^-(n)$; (12_2) $0 < f^-(n) \leqslant 1 \leqslant f^+(n)$;

while (4) clearly implies that

(13_1) $f^+(n) \geqslant f^+_k(n)$, $f^-(n) \leqslant f^-_k(n)$;

(13_2) $f - f_k = (f^- - f^-_k) f^+ + (f^+ - f^-_k) f^-_k$.

Notice that either of the functions $f^{\pm}_k$ is uniquely determined by f and k, i. e., that $(f^{\pm})_k = (f_k)^{\pm}$.

Using these notations, it will be easy to deduce from (9) the following theorem:

Every strongly multiplicative, positive function $f(n)$ *which satisfies* (8) *is almost periodic* (B); *furthermore,*

(14) $$M\{|f - f_k|\} \to 0, \text{ as } k \to \infty;$$

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[5]) E. Landau, Göttinger Nachrichten, 1900, p. 177—186; the result of I. Schur was published by I. Schoenberg, Math. Zeitschrift 28 (1928) p. 194. It should be mentioned that the corresponding result which belongs to (28) below (H. Davenport, Berliner Sitzungsberichte, 1933, p. 830—837) may also be established by the above method.

In fact, it is clear from (7) and (6) that $M\{f_k\} = c_k$. Since c^k in (6) was defined as the k-th partial product of the infinite product (9), it follows that

$$(14\text{ bis})\qquad c_k = M\{f_k\} \to M\{f\}, \text{ as } k \to \infty.$$

Hence, (14) is certainly true if either $f(n) \geqslant f_k(n)$ or $f(n) \leqslant f_k(n)$ for every n and k. It follows therefore from (13) that

$$(15)\qquad M\{|f^+ - f^+_k|\} \to 0 \text{ and } M\{|f^- - f^-_k|\} \to 0, \text{ as } k \to \infty.$$

But the function (6) of n is periodic for every f, hence also for $f^{\pm}$; so that either of the functions $f^{\pm}_k$ of n is periodic for every k. It follows therefore from (15) that either of the functions $f^{\pm}(n)$ is almost periodic (B). Since (12_2) shows that $f^-(n)$ is a bounded function, it follows from (12_1) that $f(n)$ is almost periodic (B).

In order to prove (14), notice first that, by (13_1) and (13_2),

$$(15\text{ bis})\qquad M\{|f - f_k|\} \leqslant M\{(f^-_k - f^-)f^+\} + M\{(f^+ - f^+_k)f^-\}.$$

The sum $M + M$ on the right of (15 bis) may readily be written in the form $2M\{f^-_k f^+\} - M\{f\} - M\{f_k\}$. It follows therefore from (14 bis) and (15 bis) that in order to prove (14), it is sufficient to show that $M\{f^-_k f^+\} \to M\{f\}$, as $k \to \infty$. But this is obvious from (9) and from the definitions of f^-_k and f^+.

5. The almost periodicity (B) of $f(n)$, proved in § 4, implies that the n-average $M\{f(n) \exp 2\pi i \lambda n\}$ exists for every real λ. It turns out that this Fourier coefficient vanishes for every irrational λ; so that $f(n)$ *is limit-periodic* (*grenzperiodisch*); *more explicitly, the Fourier series* (B) *of* $f(n)$ *is*

$$(16)\qquad f(n) \sim M\{f\} + M\{f\} \sum_{q>1} \sum_{m} \prod_{p|q} \frac{f(p) - 1}{f(p) - 1 + p} \cos\left(2\pi \frac{m}{q} n\right),$$

where the first (exterior) summation is over all *quadratfrei* $q > 1$, and, if q is fixed, the index p runs through all prime divisors p of q, while m through the $\Phi(q)$ values which satisfy $(m, q) = 1$ and $1 \leqslant m < q$.

In fact, (16) follows from (14), (14 bis) and (6), since P_k in (6) was defined as the product of the first k primes.

The restriction of the first summation index of (16) to *quadratfrei* $q > 1$ may be eliminated, if one introduces the Möbius function $\mu(r)$, where $r = 1, 2, 3, \ldots$. In fact, (16) may then clearly be written in the form

$$(17) \qquad f(n) \sim M\{f\} \sum_{r=1}^{\infty} \mu(r)\, c_r(n) \prod_{p|r} \frac{1-f(p)}{f(p)-1+p},$$

if $c_r(n)$ is an abbreviation for the finite sum

$$(18) \qquad c_r(n) = \sum_m \cos\left(2\pi \frac{m}{r} n\right), \text{ where } (m, r) = 1 \text{ and } 1 \leqslant m < r; \quad c_1(n) = 1.$$

Since the $\Phi(r)$ angles which occur in the sum (18) are symmetrically placed, the sum which one obtains by writing sin for cos is 0; so that

$$(18 \text{ bis}) \qquad c_r(n) = \sum_n \exp\left(2\pi i \frac{m}{r} n\right), \text{ where } (m, r) = 1 \text{ and } 1 \leqslant m < r; \quad c_1(n) = 1.$$

Thus, the $c_r(n)$ are precisely the Ramanujan sums, and so the Fourier series (B) of $f(n)$ is identical with Ramanujan's formal trigonometric series for $f(n)$. The coefficients of the series

$$(19) \qquad f(n) \sim \sum_{r=1}^{\infty} a_r c_r(n)$$

are

$$(20) \qquad a_r = a_r(f) = M\{f\}\mu(r) \prod_{p|r} \frac{1-p}{f(p)-1+p} \qquad (r = 1, 2, 3, \ldots),$$

by (17); while the expansion functions (18) of (19) may be expressed in terms of the Euler Φ-function and the Möbius μ-function as follows [6]:

$$(21) \qquad c_r(n)\, \Phi\left(\frac{r}{t}\right) = \Phi(r)\, \mu\left(\frac{r}{t}\right), \text{ where } t = (m, r);$$

(this directly implies [6]) that $c_r(n)$ is a real integer and that it represents, for fixed n, a multiplicative function of r).

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[6]) O. Hölder, Prace Mat. Fiz. 43 (1936) p. 13—23.

6. According to (16), the frequencies (Fourier exponents) of the almost periodic function $f(n)$ are rational numbers between 0 and 1 (or, rather, between -1 and 1). Let the terms of the Fourier series (16) be ordered in the Ramanujan fashion (17)–(18), and suppose that each of them actually occurs, i. e., that none of the coefficients (20) of (19) vanishes. *Then the frequencies of $f(n)$ are uniformly distributed on the tnterval* $[0,1]$ (*or, rather,* $[-1, 1]$). This may be proved as follows:

Since $|\mu(m)| \leqslant 1$, while $\Phi(m) \to \infty$ as $m \to \infty$, Hölder's formula (21) implies an observation of Ramanujan, according to which $c_r(n) = 0(1)$, when either r is fixed and $n \to \infty$, or n is fixed and $r \to \infty$. In particular

$$\text{(21 bis)} \qquad \lim_{r\to\infty} \frac{c_r(n)}{\Phi(r)} = 0 \text{ for every fixed } n \geqslant 1.$$

Now, (21 bis) is equivalent to the equidistribution of the frequencies of (19).

In fact, let $S^{(r)}$ denote, for any fixed $r \geqslant 1$, the sequence

$$\text{(22)} \qquad S^{(r)}: \frac{m_1^{(r)}}{r}, \frac{m_2^{(r)}}{r}, \ldots, \frac{m_{\Phi(r)}^{(r)}}{r}$$

of those $\Phi(r)$ fractions m/r whose numerator m satisfies the conditions $(m, r) = 1$ and $1 \leqslant m < r$. And let $\varrho_r(x)$, $0 \leqslant x \leqslant 1$, denote the distribution function of the $\Phi(r)$ fractions contained in $S^{(r)}$. Then (18 bis) shows that the ratio occurring on the left of (21 bis) is the n-th Fourier-Stieltjes coefficient of $\varrho_r(x)$, i. e., that

$$\text{(22 bis)} \qquad \int_0^1 \exp 2\pi i n x \, d\varrho_r(x) = \frac{c_r(n)}{\Phi(r)} \qquad (n \geqslant 1).$$

Thus it is clear from the criterion of Weyl for equidistribution (mod. 1), that the content of (21 bis) may be expressed as follows: The ordered infinite sequence of fractions which is obtained by writing $r = 1, 2, \ldots$ in (22) is uniformly distributed on the interval $[0, 1]$. This fact, which is equivalent to a result of Pólya may be obtained without the Fourier analysis (22 bis) of the, sequence (22) also, and contains the corresponding fact concerning the ordered infinite sequence of Farey sections [7]).

7. The considerations of (§ 4 and) § 5 may be modified in such a way as to lead from (B) to (B^2). To this end, one merely has to replace the condition (8) by the pair of conditions

$$(23) \qquad \sum \frac{|f(p)-1|}{p} < \infty, \quad \sum \frac{|f(p)^2-1|}{p} < \infty.$$

Then one obtains the following theorem:

A strongly multiplicative, positive $f(n)$ *which satisfies* (23) *is almost periodic* (B^2) *and has the Fourier expansion* (16) *or* (19), (20); *furthermore,*

$$(24) \qquad M\{(f-f_k)^2\} \to 0, \text{ as } k \to \infty,$$

and the Parseval relation takes the form

$$(25) \qquad M\{f^2\} = \sum_{r=1}^{\infty} \Phi(r)\, a_r^{\,2}.$$

In fact, if (23) is satisfied, then (4) shows that (9) is applicable to any of the three functions $f(n)^2$; $f_k(n)^2$; $f(n)\, f_k(n)$. Thus, the three averages $M\{f^2\}$; $M\{f_k^{\,2}\}$; $M\{f f_k\}$ exist and have the respective values

$$\prod_p \left(1 - \frac{1-f(p)^2}{p}\right); \quad \prod_{p \leqq p_k} \left(1 - \frac{1-f(p)^2}{p}\right);$$
$$\prod_{p \leqq p_k} \left(1 - \frac{1-f(p)^2}{p}\right) . \prod_{p > p_k} \left(1 - \frac{1-f(p)}{p}\right).$$

Hence, $M\{f(n)^2\} + M\{f_k(n)^2\} - 2M\{f(n)\, f_k(n)\} \to 0$, as $k \to \infty$. This proves (24). Since $f_k(n)$ is, by § 1, a periodic function of n, it follows from (24) that $f(n)$ is almost periodic (B^2). Finally, (25) is clear from (17), since (19) and (18) show that every amplitude (20) occurs in (17) exactly $\Phi(r)$ times.

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[7]) Cf. G. Pólya and G. Szegö, Aufgaben und Lehrsätze, Berlin, Springer 1925, chap. VIII, nos. 263—264 and chap. II, nos. 188—189.

As an illustration, consider the example (2). Then $f(p)=1-p^{-1}$; so that (23) is satisfied, and (20) shows that the coefficients (19) are

$$(26) \qquad a_r = M\{f\}\,\mu(r)\prod_{p|r}(p^2-1)^{-1}, \quad (f(n)=\Phi(n)/n).$$

8. The following theorem may be considered as the analogue to Bohr's theorem concerning uniformly almost periodic functions with linearly independent exponents:

A strongly multiplicative positive function $f(n)$ is uniformly almost periodic if and only if

$$(27) \qquad \sum |1-f(p)| < \infty .$$

The sufficiency of condition (27) is obvious, since (27) implies that the periodic functions (4) tend to $f(n)$ uniformly for all n, as $k \to \infty$. In order to prove the necessity of (27), notice first that one can assume $f(p) \leqslant 1$. In fact,

$$\prod_{(p)>1} f(p) < \infty \text{ (and therefore } \prod_{f(p)>1}(f(p)-1) < \infty),$$

since if $\prod_{f(p)>1} f(p) = \infty$, then $f(n)$ cannot be bounded. Suppose then that $f(p) \leqslant 1$ and let L be a number such that every sequence of L consecutive integers contains a translation number belonging to $1/2$. Suppose further that (27) does not hold. Then $\prod f(p) = 0$. Hence, there exist indices $k_0 = 1 < k_1 < \ldots < k_L$ satisfying

$$f(p_{k_j})f(p_{k_j+1})\ldots f(p_{k_{j+1}}) < \tfrac{1}{2} \text{ for } j=0,1,\ldots,L-1.$$

But there obviously exists an integer N such that

$$N+j \equiv 0 \pmod{p_{k_j}\,p_{k_j+1}\cdots p_{k_{j+1}}} \text{ for } j=0,1,\ldots,L-1.$$

Hence, $f(N+j) < \frac{1}{2}$ for $j=0,1,\ldots,L-1$. This contradicts $f(1)=1$, since there must exist at least one $j_0\,(0 \leqslant j_0 \leqslant L-1)$ such that $|f(N+j_0)-f(1)| < \frac{1}{2}$.

It may be mentioned that the proof could be modified in such a way as to dispose of the restriction $f(n) > 0$.

9. It is clear that all of the above considerations remain valid if one omits the condition of *strong* multiplicativity and replaces, for the resulting class of unrestricted multiplicative functions, the conditions formulated above by corresponding conditions. This holds, in particular, for the functions

$$f(n) = \frac{\sigma_\alpha(n)}{n^\alpha}, \quad (\sigma_\alpha(n) = \prod_{d|n} d^\alpha), \tag{28}$$

whose formal trigonometrical expansions (which now turn out to be Fourier expansions) were explicitly determined by RAMANUJAN.

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Рамануянові суми та майже періодичні функції

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(Резюме)

Відоме тригонометричне Рамануянове розгорнення подане без доведення одержується як розгорнення Фур'є майже періодичних функцій Безіковича. Теоретично-чисельна функція нехай буде сильно-мультиплікативна, тобто нехай має властивість (1). Виявляється, що вона тоді майже періодична і гранично-періодична. Вона має розгорнення в ряд Фур'є. Це розгорнення ідентичне розгорненню (19), де $c_r(n)$ Рамануянові суми (18 bis), а a_r визначені через (20). Функції $c_r(n)$ можна представити за допомогою Ейлерової функції Φ і Мебіусової μ, як це читаємо в формулі (21). Таким способом доконано ідентифікацію обох розгорнень.

Обліт КР 643.