ADDITIVE FUNCTIONS AND ALMOST PERIODICITY (B^2).*

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1. By an additive function $f = f(n)$ is meant a sequence $f(1), f(2), \cdots$, defined for every positive integer n in such a way that

(1) $\quad f(n_1 n_2) = f(n_1) + f(n_2)$ whenever $(n_1, n_2) = 1$; $(f(1) = 0)$.

Thus,

$$f(n) = \sum_{k=1}^{\infty} f^{(k)}(n) = \lim_{k \to \infty} f_k(n), \tag{2}$$

where $f_k = f_k(n)$ denotes, for fixed k, the additive function

$$f_k(n) = \sum_{j=1}^{k} f^{(j)}(n), \tag{3}$$

and $f^{(k)} = f^{(k)}(n)$ is the additive function which is defined in terms of the k-th prime number, p_k, as follows:

$$f^{(k)}(n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{p_k}, \\ f(p_k^{\,l}), & \text{if } p_k^{\,l} \mid n \text{ and } p_k^{\,l+1} \nmid n, \end{cases} \tag{4}$$

$(p_1 = 2, p_2 = 3, p_3 = 5, \cdots)$. Conversely, if $\{\{f(p_k^{\,l})\}\}$ is any given double sequence of numbers, then (4), (3), (2) define $f^{(k)}$, f_k, f, respectively, as additive functions of n. In fact, all but a finite number of the terms of the infinite series (2) is zero for every fixed n.

The function $f(n)$ is called multiplicative if in condition (1) the sum $f(n_1) + f(n_2)$ becomes replaced by the product $f(n_1)f(n_2)$. Conditions which are either necessary or sufficient for the almost periodicity (B^2) of a multiplicative function $f(n)$ are implied by the results of a recent paper.[1] However, none of the results found *loc. cit.*[1] supplies a criterion which is necessary and at the same time sufficient for the almost periodicity (B^2) of a multiplicative function (not even if $f(n)$ is supposed to be real-valued). This situation is not surprising, since if a real-valued multiplicative function $f(n)$ changes its sign with the uniformity of statistical randomness (as does the Möbius function $f = \mu$), then the question as to a generalized almost periodic behavior

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[1] E. R. van Kampen and Aurel Wintner, "On the almost periodic behavior of multiplicative number-theoretical functions," *American Journal of Mathematics*, vol. 62 (1940), pp. 613-626.

of $f(n)$ can involve problems of the same order of delicacy as do the relevant generalisations of the prime-number theorem, if not of the Riemann hypothesis. [While the prime-number theorem is equivalent to the statement that the n-average of $\mu(n) \exp(i\lambda n)$ exists for $\lambda = 0$, Davenport's results (*Quarterly Journal of Mathematics*, vol. 8 (1937), pp. 313-320), which were obtained by an application of the deep methods of Vinogradoff, imply that this average exists and vanishes for every real λ. In other words, all Fourier coefficients of $\mu(n)$ exist and vanish. Hence, $\mu(n)$ cannot be almost periodic (B). For if it were, the n-average of

$$|\mu(n) - (0 + 0 + \cdots)| = |\mu(n)| = |\mu(n)|^2$$

ought to vanish. But this average is known to be $6\pi^{-2} \neq 0$.]

The object of the present paper is to show that the problem admits of a definitive solution in the case of additive, instead of multiplicative, functions. In fact, the question of almost periodicity (B^2) may then completely be answered by the following theorem:

An additive function $f = f(n)$ is almost periodic (B^2) if and only if both series

$$\text{(i)} \quad \sum_p \frac{f(p)}{p}; \qquad \text{(ii)} \quad \sum_{l=1}^{\infty} \sum_p \frac{|f(p^l)|^2}{p^l}$$

are convergent.

This fact seems to be an arithmetical counterpart of a similar result concerning the case of linearly independent frequencies (cf. *loc. cit.*[3], pp. 79-80). But we were unable to find the common source of these two parallel theorems.

It is understood that $\sum_p$ denotes summation over all prime numbers, which are thought of as ordered according to magnitude (the series (i) need not be absolutely convergent).

2. If f' denotes the real, and f'' the imaginary, part of f, the function $f(n) = f'(n) + if''(n)$ is additive if and only if so are both functions $f'(n)$, $f''(n)$. Similarly, $f(n)$ is almost periodic (B^2) if and only if so are $f'(n)$ and $f''(n)$. Finally, it is clear from $|f|^2 = (f')^2 + (f'')^2$ that both series (i), (ii) are convergent if and only if so are the $2 + 2$ series which one obtains by writing f' and f'' for f in (i), (ii).

Consequently, it is sufficient to prove the italicized theorem for the case of real-valued additive functions. The possibility of this reduction is essential for the method to be applied. In fact, use will be made of a criterion which recently [2] was proved to be necessary and sufficient for those real-valued additive functions which possess an asymptotic distribution function. Now, a generalization of this criterion for complex-valued functions is not known and seems to lead to essential difficulties.

The criterion in question states [2] that a real-valued additive function $f(n)$ has an asymptotic distribution if and only if both series

$$\text{(I)}\quad \sum_p \frac{f^+(p)}{p}; \qquad\qquad \text{(II)}\quad \sum_p \frac{f^+(p)^2}{p}$$

are convergent, where $y^+ = f^+(n)$ is defined, for $y = f(n)$, by placing

$$(5)\qquad y^+ = y \text{ or } y^+ = 1 \text{ according as } |y| < 1 \text{ or } |y| \geqq 1.$$

It follows that the convergence of both series (I), (II) is necessary for every (real-valued, additive) f which is almost periodic (B^2). In fact, it is known [3] that almost periodicity in relative measure and so, in particular, almost periodicity in relative mean of any positive order ($= 2$ in the present case) is always sufficient for the existence of an asymptotic distribution function.

2 bis. Suppose, in particular, that $f(p) = O(1)$ as $p \to \infty$. Then, since

$$(6)\qquad \sum_{l=2}^{\infty} \sum_p \frac{1}{p^l} < \infty,$$

the series (ii) of § 1 is convergent if and only if so is the series

$$(7)\qquad \sum_p \frac{f(p)^2}{p};$$

hence, one readily sees from (5) that the convergence of the series (i), (ii) which occur in the criterion of § 1 is equivalent to the convergence of the respective series (I), (II) which occur in the criterion of § 2.

3. For arbitrary additive functions f, the italicized statement of § 1 will be refined by exhibiting, in case of almost periodicity (B^2), a sequence of functions which are explicitly defined in terms of f, tend to f with reference

[2] Paul Erdös and Aurel Wintner, "Additive arithmetical functions and statistical independence," *American Journal of Mathematics*, vol. 61 (1939), pp. 713-721.

[3] Börge Jessen and Aurel Wintner, "Distribution functions and the Riemann zeta function," *Transactions of the American Mathematical Society*, vol. 38 (1935), pp. 48-88, more particularly Theorem 24 (and Theorem 25).

to the metric of the space (B^2), and are almost periodic (B^2). In fact, it turns out that f *cannot be almost periodic* (B^2) *unless it is almost periodic* (B^2) *in virtue of its expansion* (2). In other words, if f is almost periodic (B^2), then, on the one hand, each of the functions $f^{(1)}, f^{(2)}, \cdots$ is almost periodic (B^2), and, on the other hand,

$$(8) \qquad M\{|f - f_k|^2\} \to 0 \text{ as } k \to \infty, \qquad (f_k = \sum_{j=1}^{k} f^{(j)}),$$

where $M\{g\} = \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} g(l)$.

3 bis. Due to this fact, it will be possible to calculate the Fourier series of f in terms of the Ramanujan sums

$$(9) \qquad c_m(n) = \Sigma'_j \exp\left(2\pi i \frac{j}{m} n\right), \text{ where } 1 \leqq j \leqq n \text{ and } (j, m) = 1.$$

In fact, the explicit form of the Fourier expansion of an arbitrary additive, almost periodic (B^2) *function* $f(n)$ *turns out to be*

$$(10) \qquad f(n) \sim a_0 + \sum_l \sum_k a_{lk} c_{p_k{}^l}(n),$$

where $l = 1, 2, 3, \cdots$, $k = 1, 2, 3, \cdots$ *and*

$$(11) \qquad a_0 = M\{f\}, \qquad a_{lk} = \sum_{i=l}^{\infty} \frac{f(p_k{}^i) - f(p_k{}^{i-1})}{p^i}.$$

Since (9) consists of $\phi(m)$ terms (ϕ = Euler's function), and since $\phi(p^l) = p^l - p^{l-1}$, the Parseval relation belonging to (10) is

$$(12) \qquad M\{|f|^2\} = |a_0|^2 + \sum_l \sum_k (p_k{}^l - p_k{}^{l-1}) |a_{lk}|^2.$$

4. It is easy to show that if f is such as to make the series (ii) of § 1 convergent, then each of the functions f_k is almost periodic (B^2).

To this end, use will be made of the following fact, proved *loc. cit.*[1] (Theorem II): If a function $g = g(n)$ of the positive integer n is such that, for some fixed prime number p, one has

$$(13) \qquad g(n) = g(p^l) \text{ whenever } p^l | n \text{ and } p^{l+1} \nmid n,$$

then g is almost periodic (B^2) if and only if

$$(14) \qquad \sum_{l=1}^{\infty} \frac{|g(p^l)|^2}{p^l} < \infty.$$

It is clear from (4) that condition (13) is satisfied by $g = f^{(k)}$ and

$p = p_k$, where k is arbitrarily fixed. Furthermore, if f is such as to make the series (ii) convergent, then, for every fixed k,

$$\sum_{l=1}^{\infty} \frac{|f(p_k{}^l)|^2}{p^l} < \infty; \tag{15}$$

so that, since $f(p_k{}^l) = f^{(k)}(p_k{}^l)$ in view of (4), condition (14) also is satisfied by $g = f^{(k)}$ and $p = p_k$. Consequently, $f^{(k)}$ is almost periodic (B^2). Since k is arbitrary, and since the almost periodic (B^2) functions form a linear space, the almost periodicity (B^2) of f_k now follows from (3).

4 bis. It was shown *loc. cit.*[1] (Theorem III) that if a function $g(n)$ satisfies (13) for some fixed prime p and is almost periodic (B), then its Fourier expansion is

$$g(n) \sim M\{g\} + \sum_l a_l c_{p^l}(n), \text{ where } a_l = \sum_{i=l}^{\infty} \frac{g(p^i) - g(p^{i-1})}{p^i}.$$

It follows therefore from § 4 that if f is such as to make the series (ii) convergent, then, for every k,

$$f^{(k)}(n) \sim M\{f^{(k)}\} + \sum_l a_{lk} c_{p_k{}^l}(n), \text{ where } a_{lk} = \sum_{i=l}^{\infty} \frac{f^{(k)}(p_k{}^i) - f^{(k)}(p_k{}^{i-1})}{p^i}.$$

Hence, (10) with (11) will follow from (4) as soon as it is proved that, on the one hand, the convergence of the series (ii) is a necessary condition for the almost periodicity (B^2) of f, and that, on the other hand, f must satisfy (8) whenever it is almost periodic (B^2).

Proof of the sufficiency of the conditions.

From here on till the end of § 9, the assumption will be that $f(n)$ is a real additive function for which both series (i), (ii) of § 1 are convergent. The final result (§ 9) will be that $f(n)$ must then be almost periodic (B^2).

5. In terms of the given $f(n)$, define an $F(n)$ as follows: $F(n)$ is that additive function for which the double sequence $\{\{F(p_k{}^l)\}\}$ is given by

$$F(p^l) = \begin{cases} f(p^l), \text{ if } |f(p)| \geqq 1; \\ f(p^l) - f(p), \text{ if } |f(p)| < 1, \end{cases} \tag{16}$$

where $p = p_k$ and $k = 1, 2, 3, \cdots$.

It is easy to see that the convergence of the series (ii) implies that

$$\sum_{l=1}^{\infty} \sum_p \frac{|F(p^l)|}{p^l} < \infty. \tag{17}$$

In fact, it is clear from (16) that the series (17) is majorized by $A + B + C$, where

$$A = \sum_p \frac{|F(p)|}{p}, \qquad B = \sum_{l=2}^{\infty} \sum_p \frac{|f(p^l)|}{p^l}, \qquad C = \sum_{l=2}^{\infty} \sum_p \frac{|f(p)|}{p^l},$$

and so it is sufficient to prove the convergence of these three series. But application of (16) to $l = 1$ shows that $F(p) = 0$ unless $|F(p)| \geqq 1$, in which case $F(p) = f(p)$; so that the series A reduces to

$$A = \sum_{|f(p)| \geqq 1} \frac{|f(p)|}{p},$$

and is therefore convergent in virtue of the assumption that the series (ii) converges. It is clear from the same assumption and from (6), that also the series B is convergent. Finally, the series C may be written in the form

$$C = \sum_{l=2}^{\infty} \sum_{|f(p)| < 1} \frac{|f(p)|}{p^l} + \sum_{l=2}^{\infty} \sum_{|f(p)| \geqq 1} \frac{|f(p)|}{p^l}.$$

But the convergence of the first of these two double series is assured by (6), while the second is, in view of

$$\sum_{l=2}^{\infty} \frac{1}{p^l} < \frac{1}{p}, \qquad (p = 2, 3, 5, \cdots),$$

majorized by

$$\sum_{|f(p)| \geqq 1} \frac{|f(p)|}{p}.$$

Since the value of the latter series was seen to be $A < \infty$, the proof of (17) is now complete.

Similarly,

$$\sum_{l=1}^{\infty} \sum_p \frac{|F(p^l)|^2}{p^l} < \infty. \tag{18}$$

In fact, since $(a - b)^2 \leqq 2(a^2 + b^2)$ for arbitrary real a, b, one sees from (16) that the series (18) is majorized by $A' + B' + C'$ where

$$A' = \sum_p \frac{|F(p)|^2}{p}, \qquad C' = 2 \sum_{l=2}^{\infty} \sum_p \frac{|f(p^l)|^2}{p^l}, \qquad B' = 2 \sum_{l=2}^{\infty} \sum_p \frac{|f(p)|^2}{p^l}.$$

And the proof for the convergence of these three series requires but a repetition (with obvious simplifications) of the above proof for the convergence of the three series A, B, C.

Notice that only the convergence of the second of the series (i), (ii) was used thus far. The same remark will hold for § 6.

6. It will now be shown that if $F_k(n)$ denotes the additive function which belongs to the additive function $F(n)$ in the same way as (3) belongs to $f(n)$, then

$$\bar{M}\{|F-F_k|^2\} \to 0 \text{ as } k \to \infty, \tag{19}$$

where $\bar{M}\{|g|^2\} = \limsup_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} |g(l)|^2$.

To this end, notice first that, by the definition (16) of the additive function $F(n)$,

$$\sum_{m=1}^{n} |F(m)-F_k(m)|^2 \leqq \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \sum_{p>k} \sum_{q>k} \frac{n}{p^l q^j} |F(p^l)F(q^j)| + \sum_{l=1}^{\infty} \sum_{p>k} \frac{n}{p^l} |F(p^l)|^2,$$

where n and k are arbitrarily fixed, and the summation indices p, q run through those primes which exceed k. On writing this inequality in the form

$$\frac{1}{n} \sum_{m=1}^{\infty} |F(m)-F_k(m)|^2 \leqq \left(\sum_{l=1}^{\infty} \sum_{p>k} \frac{|F(p^l)|}{p^l} \right)^2 + \sum_{l=1}^{\infty} \sum_{p>k} \frac{|F(p^l)|^2}{p^l},$$

keeping k fixed but letting $n \to \infty$, one sees that

$$\bar{M}\{|F-F_k|^2\} \leqq \epsilon_k^2 + \epsilon'_k, \tag{19 bis}$$

where

$$\epsilon_k = \sum_{l=1}^{\infty} \sum_{p>k} \frac{|F(p^l)|}{p^l}, \qquad \epsilon'_k = \sum_{l=1}^{\infty} \sum_{p>k} \frac{|F(p^l)|^2}{p^l}.$$

But these sums ϵ_k, ϵ'_k are identical with the k-th remainders of the convergent series (17), (18), respectively, and tend therefore to zero as $k \to \infty$. Hence, (19) is implied by (19 bis).

7. If $G_k = G_k(n)$ denotes the additive function which belongs to the additive function

$$G = f - F \tag{20}$$

in the same way as f_k, F_k belong to f, F respectively, then obviously

$$G_k = f_k - F_k. \tag{21}$$

Thus, it is clear from (16) that, for any fixed k, the elements of the double sequence $\{\{G(p^l) - G_k(p^l)\}\}$ of the additive function $G(n) - G_k(n)$ of n are independent of l, i. e., that

$$G(p) - G_k(p) = G(p^2) - G_k(p^2) = G(p^3) - G_k(p^3) = \cdots \tag{22}$$

for every prime p. It is also seen from (16) and (20) that

$$(23)\qquad | G(p) | \leqq 1$$

for every prime p.

Since the series (i) of § 1 is supposed to be convergent, it is clear from (20) and (17) that also

$$(24)\qquad \sum_p \frac{G(p)}{p} \text{ is convergent.}$$

Similarly, since the series (ii) of § 1 is supposed to be convergent, it is clear from (20), from the Schwarz inequality

$$\sum_p \frac{| f(p)F(p) |}{p} \leqq \left(\sum_p \frac{f(p)^2}{p}\right)^{\frac{1}{2}} \left(\sum_p \frac{F(p)^2}{p}\right)^{\frac{1}{2}},$$

and from (18), that

$$(25)\qquad \sum_p \frac{G(p)^2}{p} < \infty.$$

8. Due to (22), it is now easy to transcribe the O-estimates applied *loc. cit.*[2] (p. 716) into o-estimates, which are to the effect that

$$(26)\qquad \bar{M}\{| G - G_k |^2\} \to 0 \text{ as } k \to \infty.$$

In fact, (26) may be proved as follows:

If n and k are arbitrarily fixed, one readily verifies from (22) and from the definitions of the real additive functions G, G_k, that

$$\sum_{m=1}^{n} | G(m) - G_k(m) |^2 = \sum_{p>k} \sum_{q>k}{}' \left[\frac{n}{pq}\right] G(p)G(q) + \sum_{p>k} \left[\frac{n}{p}\right] G(p)^2,$$

where $[x]$ denotes the integral part of x, the prime of $\Sigma\Sigma'$ means that $p \neq q$, and the summation indices p, q run through those primes which exceed k (however, the sums on the right are finite sums for every fixed n, since

$$\left[\frac{n}{pq}\right] = 0 \text{ and } \left[\frac{n}{p}\right] = 0 \text{ whenever } pq > n \text{ and } p > n,$$

respectively). Consequently,

$$(26\text{ bis})\qquad \frac{1}{n} \sum_{m=1}^{n} | G(m) - G_k(m) |^2$$
$$\leqq \left(\sum_{k<p\leqq n^{\frac{1}{2}}} \frac{G(p)}{p}\right)^2 + 2 \sum_{n^{\frac{1}{2}}\leqq p\leqq n} \frac{G(p)}{p}\left(\sum_{k\leqq q\leqq n/p} \frac{G(q)}{q}\right)$$
$$+ \frac{1}{n} \sum_{pq\leqq n} | G(p)G(q) | + \sum_{p>k} \frac{G(p)^2}{p}.$$

8 bis. As to the inner sum in the second of the four terms on the right of (26 bis), one sees from (24) that if k is fixed and $\epsilon^{(k)}$ denotes the maximum of the function

$$\left| \sum_{k \leqq q \leqq n/p} \frac{G(q)}{q} \right|$$

of p and n on the range $n^{\frac{1}{2}} \leqq p \leqq n$; $n = 1, 2, \cdots$, then $\epsilon^{(k)} \to 0$ as $k \to \infty$; while the absolute value of the whole second term on the right of (26 bis) has, for every n, the majorant

$$2 \sum_{n^{\frac{1}{2}} \leqq p \leqq n} \frac{| G(p) |}{p} \epsilon^{(k)} \leqq 2\epsilon^{(k)} \sum_{n^{\frac{1}{2}} \leqq p \leqq n} \frac{1}{p},$$

by (23). Finally,

$$\frac{1}{n} \sum_{pq \leqq n} | G(p) G(q) | \leqq \frac{1}{n} \sum_{pq \leqq n} 1, \text{ by (23)}.$$

Thus, on keeping k fixed but letting $n \to \infty$, one sees from (26 bis) that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} | G(m) - G_k(m) |^2$$
$$\leqq \limsup_{n \to \infty} \left\{ \left(\sum_{k < p \leqq n^{\frac{1}{2}}} \frac{G(p)}{p} \right)^2 + 2\epsilon^{(k)} \sum_{n^{\frac{1}{2}} \leqq p \leqq n} \frac{1}{p} + \frac{1}{n} \sum_{pq \leqq n} 1 \right\} + \sum_{p > k} \frac{G(p)^2}{p}.$$

But p and q are prime numbers; so that

$$\sum_{n^{\frac{1}{2}} \leqq p \leqq n} \frac{1}{p} < \text{Const. and } \frac{1}{n} \sum_{pq \leqq n} 1 \to 0$$

as $n \to \infty$. Hence,

$$\bar{M}\{| G - G_k |^2\} \leqq \limsup_{n \to \infty} \left(\sum_{k < p \leqq n^{\frac{1}{2}}} \frac{G(p)}{p} \right)^2 + \text{const.}\, \epsilon^{(k)} + \sum_{p > k} \frac{G(p)^2}{p}.$$

On letting here $k \to \infty$, and using the fact $\epsilon^{(k)} \to 0$ as $k \to \infty$, one sees from (24) and (25) that the proof of (26) is complete.

9. It is now easy to conclude that $f(n)$ is almost periodic (B^2) and satisfies (8).

In fact, since it was proved in § 4 that f_k is almost periodic (B^2) in virtue of the convergence of the series (ii), it is sufficient to show that

$$\bar{M}\{| f - f_k |^2\} \to 0 \text{ as } k \to \infty.$$

But the truth of this relation is implied by (19) and (26), since it is clear from (20) and (21) that

$$\bar{M}\{| f - f_k |^2\}^{\frac{1}{2}} \leqq \bar{M}\{| F - F_k |^2\}^{\frac{1}{2}} + \bar{M}\{| G - G_k |^2\}^{\frac{1}{2}}.$$

Proof of the necessity of the conditions.

What remains to be proved is that the sufficient condition represented by the convergence of the two series (i), (ii) of § 1 is necessary as well. Thus, from now on till the end of the paper, the assumption will be that the $f(n)$ is any given real, additive function which is almost periodic (B^2).

10. Since $f(n)$ has an asymptotic distribution function, the two series (I), (II) of § 2 are convergent. And, in view of (5), the convergence of (II) implies that

$$\sum_{|f(p)|\geqq 1} \frac{1}{p} < \infty. \tag{27}$$

In terms of the given $f(n)$, define an additive function $D(n)$ by placing

$$D = f - H, \tag{28}$$

where $H = H(n)$ denotes that additive function for which the double sequence $\{\{H(p^l)\}\}$ is given by

$$H(p^l) = \begin{cases} f(p^l), & \text{if } l \neq 1, \\ f(p), & \text{if } l = 1 \text{ and } |f(p)| \geqq 1, \\ 0, & \text{if } l = 1 \text{ and } |f(p)| < 1, \end{cases} \tag{29}$$

$(p = p_k$ and $k = 1, 2, 3, \cdots)$. Thus,

$$D(p^l) = \begin{cases} 0, & \text{if } l \neq 1, \\ 0, & \text{if } l = 1 \text{ and } |f(p)| \geqq 1, \\ f(p), & \text{if } l = 1 \text{ and } |f(p)| < 1, \end{cases}$$

and so it is clear from (27) that one obtains two convergent series by writing D for f in (i)-(ii), § 1. Since the first half of the italicized statement of § 1 was already proved (§ 5–§ 9), it follows that $D(n)$ is almost periodic (B^2). Since $f(n)$ is almost periodic (B^2) by assumption, one sees from (28) that $H(n)$ is almost periodic (B^2).

In particular, $H(n)$ has a square-average

$$M\{H^2\} < +\infty. \tag{30}$$

11. In what follows, r will denote any of those prime numbers for which the absolute value of the given additive function f is not less than 1. Clearly, (27) may be written in the form

$$\prod_{|f(r)|\geqq 1} \left(1 - \frac{1}{r}\right) > 0. \tag{31}$$

Since also the density of the *quadratfrei* integers is a positive number $(= 6\pi^{-2})$, a standard application of the sieve of Erathostenes shows that (31) may be interpreted as follows: If n, j are positive integers and p is a prime, let $N = N(n, p, j)$ denote the number of those integers between 1 and n which are of the form $p^j s$, where s is *quadratfrei*, is not a multiple of p, and not a multiple of any of the primes r (defined by $|f(r)| \geqq 1$). Then there exists a constant $\beta > 0$ which is independent of n, p, j and is such that

$$N = N(n, p, j) > \beta n p^{-j}.$$

Hence, it is clear from the definition (29) of the additive function $H(n)$, that

$$\sum_{m=1}^{n} H(m)^2 > \sum_{p^l < n} \frac{\beta n}{p^l} H(p^l)^2,$$

where the summation indices $p (= 2, 3, 5, \cdots)$ and $l (= 1, 2, \cdots)$ run through those of their combinations for which $p^l < n$. Thus, on writing this inequality in the form

$$\sum_{p^l < n} \frac{H(p^l)^2}{p^l} < \text{const.} \frac{1}{n} \sum_{m=1}^{n} H(m)^2, \qquad (\text{const.} = \beta^{-1} < \infty),$$

and letting $n \to \infty$, one sees from (30) that

$$\sum_{l=1}^{\infty} \sum_{p} \frac{H(p^l)^2}{p^l} < \infty, \tag{32}$$

where p runs through all primes.

12. In view of (29), the content of (32) is that, on the one hand,

$$\sum_{l=2}^{\infty} \sum_{p} \frac{f(p^l)^2}{p^l} < \infty, \tag{33}$$

and, on the other hand,

$$\sum_{|f(p)| \geqq 1} \frac{f(p)^2}{p} < \infty; \tag{34}$$

while (34) implies that

$$\sum_{|f(p)| \geqq 1} \frac{|f(p)|}{p} < \infty. \tag{35}$$

Finally, as pointed out at the beginning of § 10 (cf. § 2), the series (I), (II) of § 2 are convergent. This means, in view of (5), that

$$\sum_{|f(p)| \leqq 1} \frac{f(p)^2}{p} < \infty \tag{36}$$

and that also

$$\sum_{|f(p)| \leqq 1} \frac{f(p)}{p} \text{ is convergent.} \tag{37}$$

Now, the convergence of the series (i) and (ii) of § 1 is clear from (37), (35) and (36), (34), (33), respectively.

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