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ON THE GAUSSIAN LAW OF ERRORS IN THE THEORY OF ADDITIVE FUNCTIONS

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In the present note we state without proofs some results concerning additive functions, the proofs of which depend partially on statistical methods. A function $f(m)$ is called additive if for $(m_1, m_2) = 1$ one has $f(m_1 \cdot m_2) = f(m_1) + f(m_2)$. We assume furthermore that $f(p^\alpha) = f(p)$ and $|f(p)| \leqslant 1$ for every prime p. None of these assumptions is essential but they simplify the statement of Theorem A.[1]

THEOREM A. *Let $f(p)$ be such that*

$$F(n) = \sum_{p < n} \frac{f^2(p)}{p}$$

diverges. Then the density of integers for which

$$f(m) < \sum_{p < m} \frac{f(p)}{p} + \omega \sqrt{2F(n)}$$

is equal to $\pi^{-1/2} \int_{-\infty}^{\omega} \exp(-y^2) dy$ *for any real* ω.

The proof depends on the following two lemmas.

LEMMA 1. *Let p_k be the kth prime and let*

$$f_k(m) = \sum_{\substack{p/m \\ p \leqslant p_k}} f(p).$$

Further let $\delta(k)$ *be the density of the integers which satisfy the inequality*

$$f_k(m) < \sum_{p \leqslant p_k} \frac{f(p)}{p} + \omega \sqrt{2 \sum_{p \leqslant p_k} \frac{f^2(p)}{p}} . \qquad (1)$$

Then

$$\lim_{k \to \infty} \delta(k) = \pi^{-1/2} \int_{-\infty}^{\omega} \exp(-y^2) dy.$$

The proof depends on the use of Fourier transforms.

LEMMA 2. *Let* $n = k^{\varphi(k)}$, *where* $\varphi(k)$ *tends to* ∞ *as* k *tends to* ∞ *arbitrarily slowly.*

Let $\psi(k, n)$ *be the number of integers* $\leqslant n$ *satisfying* (1), *and let* $\delta(k, n) = \psi(k, n)/n$.

Then

$$\lim_{k \to \infty} \delta(k, n) = \lim_{k \to \infty} \delta(k) = \pi^{-1/2} \int_{-\infty}^{\omega} \exp(-y^2) dy.$$

In order to deduce this lemma from the previous one we need Brun's method.

The proof of Theorem A now follows easily by elementary methods.[2]

From Theorem A, putting $\omega = 0$, one immediately deduces the following result:

The density of the integers which satisfy the inequality

$$f(m) < \sum_{p \leqslant m} \frac{f(p)}{p}$$

is equal to $^1/_2$.

In the special case $f(m) = \nu(m)(\nu(m)$ denotes the number of different prime divisors of m) this was proved by Erdös.[3]

[1] It suffices to assume that $\sum_{|f(p)| > 1} \frac{1}{p}$ converges.

[2] Compare P. Erdös, "On a Problem of Chowla and Some Related Problems," *Proc. Camb. Phil. Soc.*, **32,** 530–540 (1936).

[3] "Note on the Number of Prime Divisors of Integers," *Jour. Lond. Math. Soc.*, **11,** 308–314 (1936).