ON A FAMILY OF SYMMETRIC BERNOULLI CONVOLUTIONS.*

By PAUL ERDÖS.

1. For any fixed real number a in the interval $0 < a < 1$, let $\lambda = \lambda(x; a)$, $-\infty < x < +\infty$, denote the distribution function which is defined as the convolution of the infinitely many symmetric Bernoulli distribution functions $\beta(a^{-n}x)$, $-\infty < x < +\infty$, where $n = 0, 1, 2, \cdots$, and $\beta(x)$ denotes the function which is 0, $\frac{1}{2}$ or 1 according as $x < -1$, $|x| \leqq 1$ or $x > 1$. In other words, $\lambda(x; a)$ is the distribution function whose Fourier-Stieltjes transform is the infinite product

$$L(u; a) \equiv \int_{-\infty}^{+\infty} e^{iux} d_x\lambda(x; a) = \prod_{n=0}^{\infty} \cos(a^n u); \qquad -\infty < u < +\infty.$$

It is known [1] that if a value of a is not such as to make the (monotone) function $\lambda(x; a)$ of x absolutely continuous for $-\infty < x < +\infty$, then $\lambda(x; a)$ is purely singular, that is to say such as to have neither a discontinuous nor an absolutely continuous component in its Lebesgue decomposition. It is also known [2] that the set of those points of the x-axis at which the non-decreasing function $\lambda(x; a)$ is increasing either is the interval $-(1-a)^{-1} \leqq x \leqq (1-a)^{-1}$ or a nowhere dense perfect zero set contained in this interval, according as $a \geqq \frac{1}{2}$ or $a < \frac{1}{2}$. While this clearly implies that $\lambda(x; a)$ is purely singular if $a < \frac{1}{2}$, it does not imply that $\lambda(x; a)$ is absolutely continuous if $a \geqq \frac{1}{2}$. On the other hand, it is known [3] that if a has any of the values $\frac{1}{2}, (\frac{1}{2})^{1/2}, (\frac{1}{2})^{1/3}, \cdots$, then $\lambda(x; a)$ is absolutely continuous, and that $\lambda(x; (\frac{1}{2})^{1/k})$ acquires derivatives of arbitrary high order as $k \to \infty$, i. e., as $a = (\frac{1}{2})^{1/k} \to 1$.

2. Thus, one might be inclined to expect that the "smoothness" of $\lambda(x; a)$ for $-\infty < x < +\infty$ cannot decrease when a is increasing, and that $\lambda(x; a)$, being absolutely continuous if $a = \frac{1}{2}$, is absolutely continuous if $\frac{1}{2} < a < 1$, and not only if $a = (\frac{1}{2})^{1/k}$. However, it turns out that such is

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[1] B. Jessen and A. Wintner, *Transactions of the American Mathematical Society*, vol. 38 (1935), p. 61.

[2] R. Kershner and A. Wintner, *American Journal of Mathematics*, vol. 57 (1935), pp. 546-547.

[3] A. Wintner, *American Journal of Mathematics*, vol. 57 (1935), p. 837.

not the case. For instance, it will be shown that if a has the "Fibonacci" value $\frac{1}{2}(5^{1/2}-1)$, a value which lies between $\frac{1}{2}$ and $(\frac{1}{2})^{1/2}$, then $\lambda(x;a)$ is not absolutely continuous and is therefore purely singular (though nowhere constant on its range $|x| \leqq (1-\alpha)^{-1}$). A corresponding a-value between $(\frac{1}{2})^{1/2}$ and $(\frac{1}{2})^{1/3}$ is, for instance, the positive root of the cubic equation $a^3+a^2-1=0$. That $\lambda(x;a)$ is singular for these algebraic irrationalities a, will be proved by showing that the necessary condition $L(u;a)\rightarrow 0$, $u\rightarrow\pm\infty$, of the Riemann-Lebesgue lemma is not satisfied at these particular a-values.

Let α be a real algebraic integer which satisfies the inequality $\alpha>1$ and is such that, if m denotes the degree of α, and α_j, where $j=2,\cdots,m$, are the conjugates of α, then $|\alpha_j|<1$ for all j. Since $\alpha^n+\alpha_2{}^n+\cdots+\alpha_m{}^n$ is a rational integer for $n=0,1,2,\cdots$, it is clear that there exists a positive number $\theta<1$ which has the property that the distance between α^n and the nearest integer to α^n is less than θ^n for every n.

Now choose $a=1/\alpha$. Then, since $L(u;a)=\prod_{n=0}^{\infty}\cos(a^n u)$, one has, for every positive integer k,

$$L(\pi\alpha^k;a)=C\prod_{n=1}^{k}\cos(\alpha^n\pi), \text{ where } C=\prod_{n=0}^{\infty}\cos(\alpha^{-n}\pi)$$

is a non-vanishing constant, since every $\cos(\alpha^{-n}\pi)\neq 0$. Consequently by the above definition of the positive number $\theta<1$,

$$|L(\pi\alpha^k;a)|=|C|\prod_{n=1}^{k}|\cos(\alpha^n\pi)|\geqq C'\prod_{n=1}^{k}|\cos(\theta^n\pi)|,$$

where $C'=|C|\,\Pi^*|\cos\theta^n\pi|$ and the product Π^* runs through those values of n for which $\theta^n<\frac{1}{2}$. Hence, for every k,

$$|L(\pi\alpha^k;a)|\geqq C'\prod_{n=1}^{\infty}|\cos(\theta^n\pi)|=\text{const.}>0,$$

if θ is chosen to be distinct from $\frac{1}{2}$. Since $\alpha^k\rightarrow+\infty$ as $k\rightarrow+\infty$, it follows that $L(u;a)$ does not tend to 0 as $u\rightarrow\infty$, and so the distribution function $\lambda(x;a)$ is singular, for any $a=1/\alpha$ of the type described above.

It seems to be likely that these a are clustering at $a=1$ (this would imply that these a lie everywhere dense between $a=0$ and $a=1$).

3. Needless to say, $L(u;a)\rightarrow 0$, $u\rightarrow\infty$, only is a necessary condition in order that $\lambda(x;a)$ be absolutely continuous. In fact, it is known [4] that

[4] R. Kershner, *American Journal of Mathematics*, vol. 58 (1936), pp. 450-452.

if α has any rational value which is not the reciprocal value of an integer, then there exists a positive $\gamma = \gamma(a)$ such that $L(u; a) = O(|\log u|^{-\gamma})$ as $u \to \infty$, whether the positive number $a(< 1)$ is or is not greater than $\frac{1}{2}$. (It is easy to see[5] that if $a = \frac{1}{3}, \frac{1}{4}, \cdots$, then $L(u; a) \to 0$ does not hold; while $L(u; a) = (\sin u)/u$ if $a = \frac{1}{2}$.) Actually, it may be shown that, whether $a > \frac{1}{2}$ or $a < \frac{1}{2}$, the Fourier-Stieltjes transform $L(u; a)$ tends, as $u \to \infty$, to 0, not only when a is any rational number distinct from $\frac{1}{3}, \frac{1}{4}, \cdots$, but also for all irrational values of a which do not belong to a certain enumerable set.

In order to prove this, notice first that all values a between 0 and 1 which do not belong to a certain enumerable set are known[6] to possess the following property: There does not exist any number $b > 0$ in such a way that if ϵ_n denotes, for fixed a and fixed b, the distance between ba^{-n} and the nearest integer to ba^{-n}, then $\epsilon_n < \frac{1}{2}(a^{-1} + 1)^{-2}$ for every sufficiently large n. Let a be chosen such as to possess this property.

Suppose, if possible, that $L(u; a)$ does not tend to 0 as $u \to \infty$, i.e., that there exists a sequence $u_1, \cdots, u_j, \cdots$ for which one has $u_j \to \infty$ as $j \to \infty$, while $|L(u_j; a)| > c$ holds for a sufficiently small positive $c = c(a)$ which is independent of j. Clearly, one can choose these u_j in such a way that the sequence $\{b_j\}$ defined by $b_j = u_j a^k$ tends, as $j \to \infty$, to a limit, say b, where $k = k_j$ denotes the unique positive integer satisfying $a < u_j a^k \leqq 1$; so that $k_j \to \infty$ as $j \to \infty$, and $a \leqq b \leqq 1$. But $|L(u_j; a)| > c$ may be written in the form

$$|L(b_j a^{-k}; a)| = \left|\prod_{n=0}^{\infty} \cos(b_j a^{n-k})\right| > c > 0$$

for every j and $k = k_j$. Since $k_j \to \infty$ and $b_j \to b$ as $j \to \infty$, it follows by an obvious adaptation of the inequalities applied in § 2, that b has the property excluded above by the choice of a. This contradiction completes the proof of the fact that $L(u; a) \to 0$, $u \to \infty$, holds for any of the a-values under consideration.

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[5] Cf. B. Jessen and A. Wintner, *loc. cit.*, Example 1.

[6] C. Pisot, *Annali di Pisa*, ser. 2, vol. 7 (1938), p. 238.