An Extremum-Problem Concerning Trigonometric Polynomials.

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Let $S(x)$ be a trigonometric polynomial of the n^{th} order[1]) such that $|S(x)| \leqq 1$ for all real values of x. We prove that *of the graphs of all these trigonometric polynomials, those with the equations* $y = \cos(nx + \alpha)$ *(α denotes any real constant) have the maximum length of arc over* $(0, 2\pi)$.

First we need the following lemma due to VAN DER CORPUT and SCHAAKE[2]) improving upon a well known theorem of S. BERNSTEIN:

Lemma. *Let $S(x)$ be a trigonometric polynomial of the n^{th} order, such that $|S(x)| \leqq 1$. Let $T(x) = \cos nx$. Let x_1 and x_2 be two values such that*

$$-1 < S(x_1) = T(x_2) < 1,$$

then

$$|S'(x_1)| \leqq |T'(x_2)|.$$

If the sign of equality holds in a single case then it holds always, i. e. $S(x) = T(x + \alpha)$.

The following proof of the lemma[3]) is much simpler than that given by the cited authors.

Suppose the lemma be not true, i. e., although $S(x) \not\equiv T(x+\alpha)$ there is a pair of numbers x_1, x_2 such that

$$-1 < S(x_1) = T(x_2) < 1,$$
$$|S'(x_1)| \geqq |T'(x_2)|.$$

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1) "n^{th} order" stands throughout instead of "n^{th} order at most".

2) J. G. VAN DER CORPUT und G. SCHAAKE, Ungleichungen für Polynome und trigonometrische Polynome, *Compositio Math.*, **2** (1936), p. 321—361, especially Theorem 8, p. 337.

3) This proof is a generalisation of the proof of M. RIESZ for S. Bernstein's theorem, Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome, *Jahresbericht der D. M. V.*, **23** (1914), p. 354—368.

We may suppose without loss of generality that

$$x_2 = x_1, \; S'(x_1) \geqq T'(x_1) \geqq 0$$

(otherwise we should consider $S(x+\alpha)$ or $-S(x+\alpha)$ instead of $S(x)$, α being a suitable chosen real number).

First consider the case $|S(x)| < 1$, $S'(x_1) > T'(x_1)$.

Let x_1 belong to the interval $J_k = \left(\frac{k\pi}{n}, \frac{(k+1)\pi}{n}\right)$ (k odd). As

$$S\left(\frac{k\pi}{n}\right) > -1 = T\left(\frac{k\pi}{n}\right),$$

$$\left.\begin{aligned} S(x_1-\varepsilon) &< T(x_1-\varepsilon) \\ S(x_1+\varepsilon) &> T(x_1+\varepsilon) \end{aligned}\right\} \quad \text{for sufficiently small } \varepsilon,$$

$$S\left(\frac{(k+1)\pi}{n}\right) < 1 = T\left(\frac{(k+1)\pi}{n}\right),$$

the curves $y = S(x)$ and $y = T(x)$ have at least 3 points of intersection over J_k.

As the trigonometric polynomial of the n^{th} order $S(x) - T(x)$ alternates its sign in the consecutive multiples of $\frac{\pi}{n}$, it has at least $2n+2$ zeros, incongruent mod 2π, in contradiction to $S(x) \not\equiv T(x)$.

When $S(x)$ is allowed to assume the values ± 1, then our former arguments remain obviously valid if we observe that a point x where

$$S(x) = T(x) = \pm 1,$$

is at least a double zero of $S(x) - T(x)$.

Finally, if $S'(x_1) = T'(x_1)$, then x_1 is at least a double zero of $S(x) - T(x)$, so that we find also in this case more than $2n$ zeros, incongruent mod 2π. This completes the proof of the lemma.

Let us now consider an arbitrary trigonometric polynomial $S(x) \not\equiv T(x+\alpha)$ of the n^{th} order. Let σ and τ be two monotone arcs of the curves $y = S(x)$ and $y = T(x)$ respectively, the endpoints of which have the same ordinates y_1 and y_2 say. Let $|\sigma|$ and $|\sigma_x|$ denote the length of the arc σ resp. of its projection on the x-axis, $|\tau|$ and $|\tau_x|$ having analogous meaning for τ. Then we assert:

$$|\sigma| < |\tau| + (|\sigma_x| - |\tau_x|).$$

This follows easily from the lemma by approximating the arcs σ and τ by means of polygons corresponding to a subdivison of the interval (y_1, y_2).

I am indebted to Dr. P. CSILLAG for the following alternative proof: We may suppose the arcs both increasing. Writing their equations in the inverse forms $x=g(y)$, and $x=f(y)$ respectively, we deduce from the lemma that $g'(y)>f'(y)$ for $y_1<y<y_2$. Hence applying the triangle inequality to the non-degenerating triangle

$$(0, 0), \quad (1, g'(y)), \quad (1, f'(y))$$

we find

$$\{1+[g'(y)]^2\}^{1/2}<\{1+[f'(y)]^2\}^{1/2}+[g'(y)-f'(y)],$$

thus

$$|\sigma|=\int_{y_1}^{y_2}\{1+[g'(y)]^2\}^{1/2}dy<\int_{y_1}^{y_2}\{1+[f'(y)]^2\}^{1/2}dy+$$
$$+[g(y)-f(y)]_{y_1}^{y_2}=\tau+|\sigma_x|-|\tau_x|.$$

Let $\sigma', \sigma'', \ldots, \sigma^{(m)}$ $(m\geqq 2n)$ be the monotone arcs of the curve $y=S(x)$ over a suitable interval of length 2π. Denote by $\tau^{(k)}$ an arc of the curve $y=T(x)$, $0\leqq x\leqq 2\pi$, corresponding to $\sigma^{(k)}$ in the above sense. We may plainly choose the arcs $\tau', \tau'', \ldots, \tau^{(m)}$ such that no two of them overlap.

We have

$$|\sigma^{(k)}|<|\tau^{(k)}|+[|\sigma_x^{(k)}|-|\tau_x^{(k)}|]$$

whence

$$\sum_1^m|\sigma^{(k)}|<\sum_1^m|\tau^{(k)}|+\left[2\pi-\sum_1^m\tau_x^{(k)}\right].$$

On the left side we find the length of the arc $y=S(x)$, $0\leqq x\leqq 2\pi$, while the expression in brackets on the right side is the sum of the projections of the arcs remaining from the curve $y=T(x)$, $0\leqq x\leqq 2\pi$, when the arcs $\tau', \tau'', \ldots, \tau^{(m)}$ are omitted. Replacing this expression by the sum of the lengths of these additional arcs, the right side increases and becomes equal to the length of the arc $y=T(x)$, $0\leqq x\leqq 2\pi$, which concludes the proof of the theorem.

I conjecture that the following theorem holds.

Let $f(x)$ be a polynomial of the n^{th} degree, $|f(x)|\leqq 1$ in $(-1, 1)$. Of the graphs of all these polynomials that of the n^{th} Chebisheff polynomial has the maximum length of arc.

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