On definite quadratic forms, which are not the sum of two definite or semi-definite forms.

By

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Let

$$f_n = \sum_{i,j=1}^{n} a_{ij} x_i x_j \qquad (a_{ij} = a_{ji})$$

be a positive definite quadratic form with determinant D_n and integer coefficients a_{ij}. Call it an even form if all a_{ii} are even, an odd form if at least one a_{ii} is odd. Then f_n is called non-decomposable, if it cannot be expressed as a sum of two non-negative quadratic forms with integer coefficients.

Mordell [1]) proved that f_2 can always be decomposed into a sum of five squares of linear forms with integer coefficients. Ko [2]) proved that f_n can be expressed as a sum of $n+3$ integral linear squares, when $n = 3, 4, 5$.

When $n = 6$, Mordell [3]) proved that the form

$$\sum_{i=1}^{6} x_i^2 + \left(\sum_{i=1}^{6} x_i\right)^2 - 2x_1 x_2 - 2x_2 x_3 \tag{1}$$

of determinant 3 is non-decomposable; and Ko [4]) proved that (1) is the only non-decomposable form in six variables.

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[1]) Mordell, Quart. J. of Math. (Oxford) 1 (1930), 276—88.

[2]) Ko, Quart. J. of Math. (Oxford), 8 (1937), 81—98.

[3]) Mordell, Annals of Math. 38 (1937), 751—757.

[4]) May appear in Acta Arithmetica.

When $n = 7, 8$, Mordell [3]) proved that the forms

$$\sum_{i=1}^{n} x_i^2 + \Big(\sum_{i=1}^{n} x_i \Big)^2 - 2x_1 x_2 - 2x_2 x_3 \qquad (n = 7, 8)$$

with determinant $D_7 = 2$, $D_8 = 1$ are non-decomposable.

In the present paper, we shall prove the following theorems:

THEOREM 1. *When $D_n = 1$, there exists an odd non-decomposable form, if $n \geqslant 12$, except possibly for 13, 16, 17, 19, 23; and an even non-decomposable form for all $n \equiv 0 \pmod 8$.*

Hitherto the only method known for finding forms with $D_n = 1$ for $n > 8$ was that due to Minkowski [5]).

THEOREM 2. *For every $k > 0$ and $n > 13k + 176$, there exists a non-decomposable form in n variables with $D_n = k$.*

THEOREM 3. *There exist non-decomposable forms for every $n > 5$.*

From theorem 1, we can deduce that the class number h_n of positive definite quadratic forms with $D_n = 1$ is greater than $2^{\sqrt{n}}$ for large n But Magnus [6]) proved that the mass of the principal genus is greater than $n^{n^2(1-\varepsilon)/4}$ for $n > n_0$, where $\varepsilon = \varepsilon(n_0)$ is a small positive number, and so, as Dr. Mahler points out, it follows that $h_n > n^{n^2(1-\varepsilon)/4}$ for $n > n_0$.

Any quadratic form can be reduced by a unimodular transformation, i. e. integer coefficients and determinant unity, to the form

$$\sum_{i=1}^{n} a_i x_i^2 + 2 \sum_{i=1}^{n-1} b_i x_i x_{i+1} \, .$$

This and its determinant may be denoted by

$$\begin{pmatrix} a_1 & & a_2 & & \dots\, a_{n-1} & & a_n \\ & b_1 & & b_2 & \dots & b_{n-1} & \end{pmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & & a_2 & & \dots\, a_{n-1} & & a_n \\ & b_1 & & b_2 & \dots & b_{n-1} & \end{vmatrix}$$

respectively. If, however, say $a_2 = a_3 = \dots a_n = c$ and $b_2 = b_3 = \dots = b_{n-1} = d$, we may write $\begin{pmatrix} a_1 & & c_{(n-1)} \\ & b_1 & & d_{(n-2)} \end{pmatrix}$ with obviously similar extensions.

1. *Some lemmas.*

LEMMA 1. *The determinant of order n*

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[5]) Gesammelte Abhandlungen von H. Minkowski, 1, (1909), 77.

[6]) Magnus, Math. Annalen, 114 (1937), 465—475.

$$d_n = \begin{vmatrix} 2_{(n)} & \\ & 1_{(n-1)} \end{vmatrix} = n + 1.$$

It is evident that $d_1 = 2$ and $d_2 = 3$. Suppose now $d_m = m + 1$ for all $m < n$, then

$$d_n = 2d_{n-1} - d_{n-2} = 2n - (n-1) = n + 1.$$

LEMMA 2. *The only squares which can be subtracted from the form*

$$f(x) = 2 \sum_{i=1}^{n} x_i + 2 \sum_{i=1}^{n-1} x_i x_{i+1} \qquad (n > 3),$$

so that the remaining form is non-negative, are x_i^2, $(x_i + x_{i+1})^2$ $(i = 1, \ldots, n-1)$, *and* x_n^2.

Since we can write

$$f(x) = x_1^2 + \sum_{i=1}^{n-1} (x_i + x_{i+1})^2 + x_n^2.$$

the unimodular transformation

$$x_1 = y_1, \; x_i + x_{i+1} = (-1)^{i-1} y_{i+1}, \qquad (i = 1, \ldots, n-1)$$

carries $f(x)$ into

$$f(y) = \sum_{i=1}^{n} y_i^2 + (\sum_{i=1}^{n} y_i)^2.$$

If $\qquad F(y) = f(y) - (L(y))^2, \qquad L(y) = \sum_{i=1}^{n} a_i y_i$

is non-negative, then it is evident that a_i can be only ± 1 or 0 since

$$F(0, \ldots, 0, 1, 0, \ldots, 0) = 2 - a_i^2 \geqslant 0.$$

I. Suppose first that one of the a' s is zero, say $a_n = 0$. Without loss of generality, we can assume that $a_1 = \pm 1$. Then

$$F(a_1, a_2, \ldots, a_{n-1}, -a_1) = 2 + \sum_{i=2}^{n-1} a_i^2 + (\sum_{i=2}^{n-1} a_i)^2 - (1 + \sum_{i=2}^{n-1} a_i^2)^2$$

$$\leqslant 1 - \sum_{i=2}^{n-1} a_i^2 < 0,$$

if at least two of the $a_2, \ldots, a_{n-1}$ are not zero. Hence we need only consider either $a_2 = \ldots = a_{n-1} = 0$, and then $L(y) = y_1$, or only one of these a's does not vanish, say $a_2 \neq 0$. But then $F(y)$ is indefinite, since as $n > 3$,

$$F(2a_1, 2a_2, -a_1, -a_2, 0, \ldots, 0) = 2^2 + 2^2 + 1 + 1 + 2^2 - 4^2 < 0.$$

II. Suppose next that none of the a's are zero. If two of them have different sings, say $a_1 = -a_2$, then

$$F(a_1, a_2, \ldots, a_n) = n + (\sum_{i=3}^{n} a_i)^2 - n^2 \leqslant (n-2)^2 + n - n^2 < 0.$$

From I. and II, it follows that $F(y)$ is non-negative, if and only if $L(y) = y_i$ $(i = 1, \ldots, n)$, or $\sum_{i=1}^{n} y_i$. This clearly proves the lemma.

LEMMA 3. *The form*

$$f(x) = \alpha x_1^2 + 2\beta x_1 x_2 + 2 \sum_{i=2}^{n} x_i^2 + 2 \sum_{i=2}^{n-1} x_i x_{i+1}$$

with determinant $D_n < n$, where $\alpha > 0$, $\beta \geq 0$ are integers satisfying the conditions:

$$\beta^2 > \alpha > (1 - 1/n)\beta^2, \qquad 2\beta \leqslant n,$$

is positive definite and non-decomposable.

By lemma 1, $f(x)$ is positive definite, since its determinant is

$$D_n = n\alpha - (n-1)\beta^2 > 0,$$

and clearly all its principal minors are positive.

First, we shall show that nondecomposition of $f(x)$ involving a linear square exists. As in lemma 2, we can transform $f(x)$ into

$$f(y) = \alpha y_1^2 + 2\beta y_1 y_2 + \sum_{i=2}^{n} y_i^2 + (\sum_{i=2}^{n} y_i)^2.$$

By lemma 2, it follows that the only squares which need be considered are (1) $(a_1y_1)^2$, $a_1 \neq 0$, (2) $(a_1y_1+y_2)^2$, (3) $(a_1y_1+y_i)^2$ $(i=3,\ldots,n)$ and

(4) $(a_1y_1+\sum_{i=2}^{n} y_i)^2$.

The case (1) is ruled out, since $D_n - na_1^2 < 0$. For the cases (3) and (4), we need consider only the square $(a_1y_1+y_3)^2$, since $f(y)$ is symmetrical in $y_3,\ldots,y_n$, and the transformation

$$T: \qquad y_3 \to -\sum_{i=2}^{n} y_i \quad y_j \to y_j \quad (j=1,2,4,5,\ldots,n)$$

permutes y_3^2 and $(\sum_{i=2}^{n} y_i)^2$.

Consider first the form

$$F_2 = f(y) - (a_1y_1+y_2)^2$$

$$= (\alpha - a_1^2)\, y_1^2 + 2(\beta - a_1)y_1y_2 + \sum_{i=3}^{n} y_i^2 + (\sum_{i=2}^{n} y_i)^2.$$

The transformation

$$y_2 \to -\sum_{i=1}^{n} y_i, \quad y_j \to y_j \quad (j=1,3,4,\ldots,n)$$

carries F_2 into

$$F_2' = (\alpha - a_1^2)y_1^2 - 2(\beta - a_1)\, y_1(\sum_{i=2}^{n} y_i) + \sum_{i=2}^{n} y_i^2$$

$$= \sum_{i=2}^{n} (y_i - (\beta - a_1)\, y_1)^2 + (\alpha - a_1^2 - (n-1)(\beta - a_1)^2)y_1^2.$$

The maximum of the coefficients of y_1^2

$$A = \alpha - a_1^2 - (n-1)(\beta - a_1)^2$$

for different a_1 occurs when $a_1 = (n-1)\,\beta/n$. Since $0 < \beta/n < 1$, we have for $a_1 = \beta$, $\beta - 1$, respectively,

$$A=\alpha-\beta^2<0 \quad \text{and} \quad A=\alpha-\beta^2+2\beta-n<0,$$

so that F_2' is indefinite. This settles the case (2).

Consider next the form

$$F_3=f(y)-(a_1 y_1+y_3)^2$$

$$=(\alpha-a_1^2)\,y_1^2+2\beta y_1 y_2+y_2^2+\sum_{i=4}^{n} y_i^2+(\sum_{i=2}^{n} y_i)^2-2a_1 y_1 y_3.$$

The transformation T carries F_3 into

$$F_3'=(\alpha-a_1^2)\,y_1^2+2\beta y_1 y_2+2a_1 y_1(\sum_{i=2}^{n} y_i)+\sum_{i=2}^{n} y_i^2$$

$$=\sum_{i=3}^{n}(y_i+a_1 y_1)^2+(y_2+\beta+a_1)\,y_1)^2+(\alpha-a_1^2-(\beta+a_1)^2-(n-2)a_1^2)y_1^2.$$

The maximum value of the coefficient of y_1^2

$$A'=\alpha-a_1^2-(\beta+a_1)^2-(n-2)a_1^2$$

is reached when $a_1=-\beta/n$. Since $-1<-\beta/n<0$, we have, for $a_1=0,-1$, respectively,

$$A'=\alpha-\beta^2<0 \quad \text{and} \quad A'=\alpha-\beta^2+2\beta-n<0,$$

F_3 is indefinite and cases (3) and (4) are also settled.

Suppose now there is a decomposition

$$f(x)=f'(x)+f''(x).$$

No term x_i^2 $(i\geq 2)$ can occur in either $f'(x)$ or $f''(x)$ for then a square can be taken out of $f(x)$. Hence we can assume $f'(x)$, say, has a term $2x_n^2$. Then $f'(x)$ must also contain $2x_{n-1}x_n$, for otherwise $f''(x)$ assumes negative values by choice of x_n. Then $f'(x)$ contains also $2x_{n-1}^2$, for otherwise $f'(x)$ will assume negative values by choice of x_{n-1}. Proceeding in this way, $f'(x)$ will contain all the terms of $f(x)$ involving $x_n, x_{n-1}, \ldots, x_2$. Hence $f''(x)=ax_1^2$, and so a square x_1^2 can be taken out from $f(x)$, which contradicts what we hawe proved.

LEMMA 4. *If $n\neq 2^\alpha$, p^α, $2p^\alpha$, where p is an odd prime and α is a positive integer, then there exists an odd non-decomposable form in n variables with determinant unity.*

Consider the form

$$f_n = \begin{pmatrix} x & 2_{(n-1)} & \\ & y & 1_{(n-2)} \end{pmatrix}$$

in n variables. It is easy to calculate by using lemma 1 that its determinant has the value

$$D_n = nx - (n-1)y^2.$$

Putting $D_n = 1$, we have to solve the congruence

(2) $$y^2 \equiv 1 \pmod{n}.$$

Since $n \neq 2^\alpha,\ p^\alpha,\ 2p^\alpha$, we can write

$$n = a \cdot b,\ (a, b) = 1,\ a > 2, \text{ and } b > 2.$$

Suppose y_1, y_2 are the solutions of the congruences:

$$y_1 \equiv -1 \pmod{a},\ y_1 \equiv 1 \pmod{b},\ 0 < y_1 < n;$$
$$y_2 \equiv 1 \pmod{a},\ y_2 \equiv -1 \pmod{b},\ 0 < y_2 < n.$$

Both y_1 and y_2 satisfy the congruence (2) and since

$$y_1 + y_2 \equiv 0 \pmod{n},\ 0 < y_1 < n,\ 0 < y_2 < n,$$

we have

$$y_1 + y_2 = n.$$

Hence one of the y_1, y_2 is less than $\frac{1}{2}n$ and we take this value to be our y, which satisfies the inequality $2y < n$.

From $D_n = 1$, we can obtain the inequalities $y^2 > x > (1 - 1/n)y^2$. Hence the form f_n satisfies all the conditions of lemma 3 and is non-decomposable.

f_n is an odd form if $x = ((n-1)y^2 + 1)/n$ is odd x is evidently odd if n is odd. If n is even, we write

$$x = y^2 - (y^2 - 1)/n.$$

Then y must be odd and from the congruences

$$y \equiv \pm 1 \pmod{a},\ y \equiv \mp 1 \pmod{b},\ (a, b) = 1,\ ab = n,$$

it is clear that if a is even, then b is odd, $y \pm 1$ is even and so $(y^2 - 1)/n$ is even and so x is odd.

LEMMA 5. *For any $n = 8m$, there exists an even non-decomposable form in n variables with determinant unity.*

Consider the form

$$f(x) = \begin{pmatrix} 8m & & 2m & & 2_{(8m-2)} \\ & 4m-1 & & 1_{(8m-2)} & \end{pmatrix}$$

in $8m$ variables. By lemma 1, the right lower corner $(8m-1)$-rowed minor or the determinant D_{8m} of $f(x)$ has the value

$$2m(8m-1)-(8m-2)=16m^2-10m+2>0,$$

and so

$$D_{8m}=8m(16m^2-10m+2)-(4m-1)^2(8m-1)=1.$$

Hence it is clear that $f(x)$ is an even positive definite quadratic form with determinant unity.

To prove the non-decomposability of $f(x)$, we first show that no square can be taken out from $f(x)$.

Let Q be the matrix of $f(x)$, then the adjoint form of $f(x)$, say F, has matrix Q^{-1}. Since

$$QQ^{-1}Q=Q,$$

$F \sim f(x)$, and so F is also even. Hence all the $(8m-1)$-rowed minors of forms equivalent to $f(x)$ are even. Suppose now a square L^2 can be taken out from $f(x)$. A unimodular transformation carries $f(x)$ into

$$f'(x)=\sum_{i,j=1}^{8m} a_{ij}x_ix_j \qquad (a_{ij}=a_{ji})$$

and $L=x_1$. Then the determinant of $f(x)-x_1^2$ is

$$\begin{vmatrix} a_{11}-1 & a_{12} & \dots a_{1,8m} \\ a_{21} & a_{22} & \dots a_{2,8m} \\ \dots & \dots & \dots\dots \\ a_{8m,1} & a_{8m,2} & \dots a_{8m,8m} \end{vmatrix} = 1-A,$$

where A is the minor of the element a_{11} in the determinant of $f(x)$. Since A is even, $1-A<0$ and so $f'(x)-x_1^2$ is indefinite.

Suppose now $f(x)$ is decomposable, say

$$f(x)=f_1(x)+f_2(x).$$

By the same argument used in the last part of the proof of lemma 3, one of the $f_1(x), f_2(x)$, say $f_2(x)$, can at most contain the variables x_1 and x_2. Since all binary non-negative forms can be expressed as a sum of squares of linear forms, a square can be taken out from $f(x)$. This contradicts what we have just proved [7]).

LEMMA 6. *If there exists an even positive form in n variables with determinant unity, then n is divisible by* 8.

Suppose there exists an even form f_n with determinant $D_n = 1$. Then by a unimodular transformation, we can change f_n into

$$\begin{pmatrix} 2a_1 & & 2a_2 & & 2a_3 & & \ldots 2a_{n-1} & & 2a_n \\ & b_1 & & b_2 & & b_3 & \ldots & b_{n-1} & \end{pmatrix}.$$

A simple determinant calculation shows that D_n is even if n is odd. Hence n is even. Let the left hand corner principal minors of D_n be $2D_1$, D_2, $2D_3$, $D_4, \ldots, 2D_{n-1}$, and write $D_0 = 1$, then

$$\begin{aligned} 2D_1 &= 2a_1, D_2 = 4a_2 D_1 - D_0 b_1^2, \ldots, D_{2i-1} = a_{2i-1} D_{2i-2} - D_{2i-3} b_{2i-2}^2, \\ D_{2i} &= 4a_{2i} D_{2i-1} - D_{2i-2} b_{2i-1}^2, \ldots, D_n = 4a_n D_{n-1} - D_{n-2} b_{n-1}^2 = 1. \end{aligned} \tag{3}$$

From these relations, it is easy to see that $(D_i, D_{i+1}) \mid D_n = 1$ and so $(D_i, D_{i+1}) = 1$ for $i = 1, \ldots, n-1$. Since

$$D_n \equiv \begin{vmatrix} 0 & & 0 & & 0 & \ldots 0 & & 0 \\ & b_1 & & b_2 & & b_3 \ldots b_{n-1} & \end{vmatrix} \equiv b_1 b_3 \ldots b_{n-1} \pmod 2,$$

all the b_{2i+1} are odd. By taking congruences modulus 4 in (3), we have

$$D_2 \equiv - b_1^2 \equiv -1, \qquad D_4 \equiv - D_2 b_3^2 \equiv 1 \pmod 4.$$

It follows, by induction, that in general

$$D_{4i+2} \equiv -1 \qquad \text{and} \qquad D_{4i} \equiv 1 \pmod 4.$$

Hence the D_{2i} are odd and $n \equiv 0 \pmod 4$, say $n = 4m$. Write $D_{2i+1} = 2^{t_{2i+1}} D'_{2i+1}$, where D'_{2i+1} is odd. It is evident from the last relation of (3), that the D'_{4m-1}, D_{4m-2} satisfy the relation

[7]) This argument shows that the even positive definite form with determinant unity

$$h(x) = \sum_{i=1}^{8m} x_i^2 + (\sum_{i=1}^{8m} x_i)^2 - 2x_1x_2 - 2x_2x_{8m} + 2(m-1)x_{8m}^2$$

given by Korkine and Zolotareff in Mathematische Annalen, 6, 1873, p. 366—389 (brought to our notice by prof. L. J. Mordell) is non-decomposable. It is probable that $h(x)$ is equivalent to our $f(x)$ for the same m.

$$(-D_{4m-2}/D'_{4m-1}) = 1,$$

the symbol being that of quadratic residuacity. Since $-D_{4m-2} \equiv 1 \pmod 4$, and $D_{4m-2} \equiv -1 \pmod 8$, when $t_{4m-1} > 0$, we have

$$1 = (D'_{4m-1}/D_{4m-2}) = (D_{4m-1}/D_{4m-2}).$$

From the relation $D_{4m-1} = a_{4m-1} D_{4m-2} - D_{4m-3} b^2_{4m-2}$ of (3),

$$\begin{aligned} 1 &= (-D_{4m-3}/D_{4m-2}) \\ &= (2^{t_{4m-3}}/D_{4m-2}).(-1)^{\frac{1}{2}(D'_{4m-3}+1)} (D_{4m-2}/D'_{4m-3}) \\ &= (2^{t_{4m-3}}/D_{4m-2}).(-1)^{\frac{1}{2}(D'_{4m-3}+1)} (-D_{4m-4}/D'_{4m-3}), \end{aligned}$$

since again from (3), $D_{4m-2} = 4a_{4m-2} D_{4m-3} - D_{4m-4} b^2_{4m-3}$. Hence

$$\begin{aligned} 1 &= (2^{t_{4m-3}}/D_{4m-2}).(-1)^{\frac{1}{2}(D'_{4m-3}+1)+\frac{1}{2}(D'_{4m-3}-1)} (D'_{4m-3}/D_{4m-4}) \\ &= -(2^{t_{4m-3}}/D_{4m-2})(2^{t_{4m-3}}/D_{4m-4})(D_{4m-3}/D_{4m-4}). \end{aligned}$$

From the relation $4a_{4m-2} D_{4m-3} - D_{4m-4} b^2_{4m-3} = D_{4m-2}$, we have, when $t_{4m-3} > 0$, since b_{4m-3} is odd, $D_{4m-2} + D_{4m-4} \equiv 0 \pmod 8$. Hence

$$(2/D_{4m-2}) = (2/D_{4m-4}) \text{ and so}$$
$$(2^{t_{4m-3}}/D_{4m-2})(2^{t_{4m-3}}/D_{4m-4}) = 1.$$

Hence

$$1 = -(D_{4m-3}/D_{4m-4}),$$

or

$$(D_{4m-3}/D_{4m-4}) = -1.$$

Continuing this process, we get

$$(D_{4m-8i-3}/D_{4m-8i-4}) = -1.$$

Hence

$$D_{4m-8i-4} \neq 1,$$

and so $4m - 8i - 4 \neq 0$, or n is divisible by 8.

LEMMA 7. *The positive definite forms:*

$$f_{8m-1} = \begin{pmatrix} 8m & 2m & 2_{(8m-3)} \\ \cdot & 4m-1 & 1_{(8m-3)} \end{pmatrix},$$

$$f_{8m-2} = \begin{pmatrix} 8m & 2m & 2_{(8m-4)} \\ & 4m-1 & 1_{(8m-4)} \end{pmatrix},$$

in $8m-1$ and $8m-2$ variables with determinants 2 and 3, respectively, are non-decomposable.

Let us first consider the form f_{8m-2}. From the argument used in the last part of the proof of lemma 5, it suffices to prove that no square can be subtracted from f_{8m-2}. Suppose $f_{8m-2}-L^2$ is a non-negative quadratic form with integer coefficients, where L is a linear form in $x_1, \ldots, x_{8m-2}$ with integer coefficients having no common factor. By an unimodular transformation, we can write $L = x_1$, and then

$$f_{8m-2} \sim f'_{8m-2} = \sum_{i,j=1}^{8m-2} a_{ij} x_i x_j \qquad (a_{ij} = a_{ji}),$$

where $f'_{8m-2} - x_1^2$ is a non-negative form. Let the cofactor of a_{11} in the determinant of f'_{8m-2} be A_{21}; then the determinant of $f'_{8m-2} - x_1^2$ is $3 - A_{11}$ and ist not negative. Since the adjoint form of an even form in an even number of variables is even [8]), $A_{11} = 2$. Consider now the positive even definite form

$$f_{8m+4} = 8x_1^2 + 6x_1x_2 + 2\sum_{i=2}^{6} x_i^2 + 2\sum_{i=2}^{6} x_i x_{i+1} + \sum_{i,j=1}^{8m-2} a_{ij} x_{i+6} x_{j+6}$$

in $8m+4$ variables. On bearing in mind the method of lemma 1, the lower right corner, say l. r. c., $(8m-1)$-rowed minor of the determinant of f_{8m+4} has the value $2.3-2=4$; the l. r. c. $8m$-rowed minor is $2.4-3=5$, the l c. r. $(8m+1)$-rowed minor is $2.5-4=6$, the l r. c. $(8m+2)$-rowed minor is $2.6-5=7$, the l. r. c. $(8m+3)$-rowed minor is $2.7-6=8$ and so the determinant of f_{8m+4} is $8.8-3^2.7=1$, which contradicts lemma 6.

Next we prove that no square can be taken out from f_{8m-1} and hence f_{8m-1} is non-decomposable. If $f_{8m-1}-L^2$ is non-negative, then L cannot contain a term involving x_i $(1 \leqslant i \leqslant 8m-2)$, for otherwise, by putting $x_{8m-1}=0$, we would get a decomposition of f_{8m-2}. Hence $L = x_{8m-1}$. But $f_{8m-1} - x_{8m-1}^2$ is indefinite, since the determinant of $f_{8m-1} - x_{8m-1}^2$ is $2-3<0$. This completes the proof.

LEMMA 8. *Let the positive definite quadratic forms:*

$$g_1 = f_m(x_1, \ldots, x_m), \qquad g_2 = f_{n-m-1}(x_{m+2}, \ldots, x_n),$$
$$g_3 = bx_{m+1}^2 + 2x_{m+1}x_{m+2} + g_2$$

having determinants $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$, respectively, be non-decomposable. Denote

[8]) Bachmann, Zahlentheorie, vol. 4, part 1, 444.

by A the value of the upper left-hand corner principal $(m-1)$-rowed minor of $\mathfrak{D}_1$. If there exists a positive definite quadratic form of determinant $\mathfrak{D} < \mathfrak{D}_1 \mathfrak{D}_2$.

$$g = g_1 + ax_{m+1}^2 + 2x_m x_{m+1} + g_3,$$

where a is an integer and $0 < a < A/\mathfrak{D}_1$, then g is non-decomposable.

Suppose g has a decomposition

$$g = h + h'. \tag{4}$$

If one of the h's has a term involving x_i $(i = 1, \ldots, m)$, it will contain all the terms of g_1, for otherwise we would get a decomposition of g_1 by putting $x_{m+1} = \ldots = x_n = 0$. Similarly, if one of the h's, say h, has a term involving x_i $(i = m+2, \ldots, n)$, it contains all the terms of g_2. Then h must contain the term $2x_{m+1} x_{m+2}$, for otherwise, h' will assume negative values by choice of x_{m+2}. Then h contains also a term $b'x_{m+1}^2$ with $b' > 0$, for otherwise, h will assume negative values by choice of x_{m+1}. Next $b' \geq b$, for if $b' < b$, on putting $x_1 = \ldots = x_m = 0$,

$$h = g_2 + 2x_{m+1}x_{m+2} + b' x_{m+1}^2.$$

This is indefinite, since g_3 is non-decomposable. Hence h contains g_3. Hence we may suppose that either h contains both g_1 and g_3, or h contains g_1 and h' contains g_3.

In the first case, h' can only contain the terms or part of the terms of $g - (g_1 + g_3) = ax_{m+1}^2 + 2x_m x_{m+1}$. Then $h' = cx_{m+1}^2$ with $0 < c \leq a$, since if h' contains the $2x_m x_{m+1}$, h' will assume negative values by choice of x_m. Hence

$$h = g - cx_{m+1}^2.$$

Since the cofactor of the coefficients of x_{m+1}^2 in the determinant $\mathfrak{D}$ of g, is $\mathfrak{D}_1 \mathfrak{D}_2$, the determinant of h is $\mathfrak{D} - c\,\mathfrak{D}_1 \mathfrak{D}_2$. By hypothesis, $\mathfrak{D} - \mathfrak{D}_1 \mathfrak{D}_2 < 0$, h is indefinite.

In the second case, h must contain the term $2x_m x_{m+1}$, for otherwise h' will assume negative values by choice of x_m. Then h contains also a term $c'x_{m+1}^2$, $c' > 0$, for otherwise h will assume negative values by choice of x_{m+1}. Also

$$h = g_1 + 2x_m x_{m+1} + c'x_{m+1}^2, \quad h' = g_3 + dx_{m+1}^2,$$

since h' contains g_3. Hence $a = c + d$ and so $c \leq a$ for d cannot be negative, as g_3 is indecomposable. It is easy to see that the determinant of h is $c'\,\mathfrak{D}_1 - A$. By hypothesis, $c'\,\mathfrak{D}_1 \leq a\,\mathfrak{D}_1 \leq A$, and so h is indefinite. Hence (4) is impossible and the lemma is proved.

LEMMA 9. *For every $n \geqq 12$, except possibly for $n = 13, 16, 17, 19, 23$, there exists an odd non-decomposable quadratic form with determinant unity.*

Suppose $n+2$ can be expressed as the sum of two positive integers n_1, n_2, where $n_1 = 8m$ or $a^2 - 1$, and $n_2 \neq 4, p^\alpha, 2p^\alpha$, p being an odd prime and α an integer. Let the form

$$(5) \qquad \begin{pmatrix} a_1 & & a_2 & & a_3 & & \dots a_{n_1-2} & & a_{n_1-1} \\ & b_1 & & b_2 & & b_3 & \dots & b_{n_1-2} & \end{pmatrix}.$$

in $n_1 - 1$ variables with determinant 2, containing a minor

$$(6) \qquad \begin{vmatrix} a_1 & & a_2 & & a_3 & & \dots a_{n_1-3} & & a_{n_1-2} \\ & b_1 & & b_2 & & b_3 & \dots & b_{n_1-3} & \end{vmatrix} = 3$$

be non-decomposable. Such forms always exist, for if $n_1 = 8m$, we can by lemma 7, take the form

$$\begin{pmatrix} 8m & & 2m & & 2_{(8m-3)} \\ & 4m-1 & & 1_{(8m-3)} & \end{pmatrix};$$

and if $n_1 = a^2 - 1$, by lemma 3, the form

$$\begin{pmatrix} a^2-1 & & 2_{(n_1-2)} & \\ & a & & 1_{(n_1-3)} \end{pmatrix}.$$

Consider now the odd form:

$$f_n = \begin{pmatrix} a_1 & & a_2 & & a_3 & & \dots a_{n_1-2} & & a_{n_1-1} & & 3 & & 2_{(n_2-3)} & & x \\ & b_1 & & b_2 & & b_3 & \dots & b_{n_1-2} & & & & 1 & & 1_{(n_2-3)} & & y \end{pmatrix},$$

in n variables with x, y satisfying the relation

$$(7) \qquad n_2 x - (n_2 - 1) y^2 = 1.$$

From (5) and (6),

$$\begin{vmatrix} a_1 & & a_2 & & a_3 & & \dots a_{n_1-2} & & a_{n_1-1} & & 3 \\ & b_1 & & b_2 & & b_3 & \dots & b_{n_1-2} & & & & 1 \end{vmatrix} = 3.2 - 3 = 3,$$

and so as in lemma 1,

$$\begin{vmatrix} a_1 & & \dots a_{n_1-2} & & a_{n_1-1} & & 3 & & 2(n_2-4) \\ & b_1 & \dots & b_{n_1-2} & & & & 1 & & 1(n_2-4) \end{vmatrix} = n_2 - 1,$$

$$\begin{vmatrix} a_1 & & \dots a_{n_1-2} & & a_{n_1-1} & & 3 & & 2(n_2-3) \\ & b_1 & \dots & b_{n_1-2} & & & & 1 & & 1(n_2-3) \end{vmatrix} = n_2,$$

and the determinant of f_n is $n_2 x - (n_2 - 1) y^2 = 1$. From lemma 8, on taking

$$g_1 = \begin{pmatrix} a_1 & \cdots & a_{n_1-2} & & a_{n_1-1} \\ & b_1 & \cdots & b_{n_1-2} & \end{pmatrix}, \quad \mathcal{D}_1 = 2,\ A = 3,$$

$$g_2 = \begin{pmatrix} 2^{(n_2-4)} & & 2 & & x \\ & 1^{(n_2-4)} & & y & \end{pmatrix},\ a = 1,\ g_3 = \begin{pmatrix} 2^{(n_2-3)} & & 2 & & x \\ & 1^{(n_2-3)} & & y & \end{pmatrix},$$

$$\mathcal{D}_2 = \begin{vmatrix} 2^{(n_2-4)} & & 2 & & x \\ & 1^{(n_2-4)} & & y & \end{vmatrix},\ \mathcal{D}_3 = \begin{vmatrix} 2^{(n_2-3)} & & 2 & & x \\ & 1^{(n_2-4)} & & y & \end{vmatrix};$$

f_n is non-decomposable if g_2, g_3 are non-decomposable. From lemma 3, we need only show that

(8) $$x < y^2,$$

(9) $$2y \leqslant n_2 - 2,$$

the determinant of order $n_2 - 2$,

(10) $$\mathcal{D}_2 < n_2 - 2,$$

and the determinant of order $n_2 - 1$

(11) $$\mathcal{D}_3 < n_2 - 1.$$

By lemma 1 and (7),

(12) $$\begin{aligned} \mathcal{D}_2 &= (n_2 - 2)x - (n_2 - 3)y^2 = 1 - 2x + 2y^2, \\ \mathcal{D}_3 &= (n_2 - 1)x - (n_2 - 2)y^2 = 1 - x + y^2. \end{aligned}$$

We now solve (7). Since $n_2 \neq 4, p^\alpha, 2p^\alpha$,

$$Y^2 \equiv 1 \pmod{n_2}$$

has a solution Y satisfying the inequalities

(13) $$1 < Y < \tfrac{1}{2} n_2.$$

Then taking $y = Y$ in (7), we have a solution (x, y). Then (8) evidently holds, as from (7) and (13),

$$x = y^2 + (1 - y^2)/n_2 < y^2.$$

If n_2 is even, (9) follows from (13). If n_2 is odd, say $n_2 = 2n_3 + 1$, $y \neq n_3$, since $n_3^2 \not\equiv 1 \pmod{2n_3 + 1}$ for $n_2 \neq 3$. Hence from (13), $y \leqslant n_3 - 1$ and (9) holds again. Since from (12), $\mathcal{D}_2 - \mathcal{D}_3 = y^2 - x > 0$, (11) holds if (10) holds. From (7) and (9), we get

$$y^2 - x = (y^2 - 1)/n_2 < n_2/4.$$

Thus (10) follows if $n_2 \geqq 6$, since

$$\mathcal{D}_2 = 1 + 2y^2 - 2x < 1 + n_2/2 \leqslant n_2 - 2$$

is true for $n_2 \geqq 6$. But $n_2 \neq 4, p^\alpha, 2p^\alpha$, and so $n_2 \geqq 8$, hence f_n is non-decomposable.

Now from lemma 4, we need only prove that if $n = 2^k$, p^k, or $2p^k$, where $n \geqq 12$, $n \neq 13, 16, 17, 19, 23$, the equation

$$n + 2 = n_1 + n_2$$

is solvable with the conditions $n_1 = 8m$ or $a^\gamma - 1$, $n_2 \neq 4$, p^α, $2p^\alpha$ and $n_1 > 0$, $n_2 > 0$.

Small values for $n_1 - 2$ are

$$6, 13, 14, 22, 30, 38 \text{ and } 46.$$

Suppose first that $n \equiv 0 \pmod 4$. Then we need only consider $n = 2^k$.

If $2^k \equiv 2 \pmod 3$, then we can take $n_2 = 2^k - 14$ or $2^k - 38$, if $n > 38$, unless

$$2^k - 14 = 2.3^\beta, \qquad 2^k - 38 = 2.3^\gamma.$$

They give $3^\beta - 3^\gamma = 12$, which is impossible. But if $n \leqslant 38$, we get the exceptional case $n = 32$.

If $2^k \equiv 1 \pmod 3$, then we can take $n_2 = 2^k - 22$ or $2^k - 46$, if $n > 46$, unless

$$2^k - 22 = 2.3^\beta, \qquad 2^k - 46 = 2.3^\gamma.$$

They give also the impossible equation $3^\beta - 3^\gamma = 12$ and we get the exceptional value $n = 16$.

Suppose next $n \equiv 2 \pmod 4$, we can take $n_2 = n - 6$, unless $n_2 = 4$, i. e. $n = 10$.

Suppose finally n is odd and so $n = p^k$. If $n \equiv 0 \pmod 3$, we can take $n_2 = n - 6$ or $n - 30$, if $n > 30$, unless

$$n - 6 = 3^\beta, \qquad n - 30 = 3^\gamma.$$

They give the equation $3^\beta - 3^\gamma = 24$, which has only the solutions $\beta = 3$ leading to $n = 33 \neq p^k$ The only exceptional value $n = p^k \leqslant 30$ is 27.

If $n \equiv 2 \pmod 3$, we can take $n_2 = n - 14$, or $n - 38$, if $n > 38$, unless

$$n - 14 = 3^\beta, \qquad n - 38 = 3^\gamma.$$

They give the equation $3^\beta - 3^\gamma = 24$, which has the only solution $\beta = 3$ and this corresponds $n = 41$. The other exceptional values $\leqslant 38$ are 17, 23, 29.

If $n \equiv 1 \pmod 3$, we can take $n_2 = n - 22$ or $n - 13$, if $n > 22$, unless

$$n-22=3^{\beta}, \quad n-13=3^{\gamma}.$$

They give the impossible equation $3^{\gamma}-3^{\beta}=9$, and so the exceptional values in this case are only 13, 19.

Hence the exceptional values are

$$n=13, 16, 17, 19, 23, 27, 29, 32 \text{ and } 41.$$

Since

$$27-6=21, \quad 29-14=15, \quad 41-6=35,$$

and 21, 15, $35 \neq 4, p^{\alpha}, 2p^{\alpha}$, we can rule out the cases 27, 29 and 41. Hence the only exceptional values are

$$n=13, 16, 17, 19, 23 \text{ and } 32.$$

But 32 can be excluded from the last. Write

$$f_{31}=\begin{pmatrix}35 & & 2(30) & \\ & 6 & & 1(29)\end{pmatrix}, \quad f_{32}=\begin{pmatrix}35 & & 2(29) & & 2 & & 5\\ & 6 & & 1(29) & & 2 & \end{pmatrix}.$$

Then f_{31} has determinant $5=35 \cdot 31-6^2 \cdot 30$, f_{32} has determinant $1=5 \cdot 5-2^2(35 \cdot 30-6^2 \cdot 29)$. By lemma 3, the form f_{31} is non-decomposable. If there exists a decomposition for f_{32}, say

$$f_{32}=h_{32}+h_{32}'$$

and one of the h's, say h_{32} must vanish identically if we put $x_{32}=0$, for otherwise, there would exist a decomposition for f_{31}. Hence h_{32}' contains only cx_{32}^2 with $c \geqq 1$. This is impossible, since

$$\begin{vmatrix}35 & & 2(29) & & 2 & & 5-c\\ & 6 & & 1(29) & & 2 & \end{vmatrix}=1-5c<0.$$

Hence f_{32} is non-decomposable and our lemma is proved.

It should be remarked that for $n=8$ [9]), 9, 10, 11, 13 [10]), it is known that there exist no odd non-decomposable forms with determinant unity. It still remains to be investigated whether there exist odd non-decomposable forms when $n=16$, 17, 19 and 23 with determinant unity.

LEMMA 10. *For every odd integer $n>176$, a non-decomposable form in n variables with determinant 2 exists such that the upper left-hand*

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[9]) Mordell, J. de Mathématiques, 17 (1938), 41—46. Also see Ko, Quart. J. of Math. (Oxford), 8 (1937) 85.

[10]) Ko, „On the positive definite quadratic forms with determinant unity", which may appear in Acta Arithmetica.

$(n-1)$-rowed principal minor of its determinant is odd and greater than unity.

We prove first the existence of two such forms in $16k+1$, $22h+1$ variables respectively.

Consider first the form in $16k+1$ variables:

$$(14)\quad f_{16k+1}=\begin{pmatrix} 2(15) & & 2 & & 34 & & 10 & & 2(14) & & 34 & & 10 & & 2(14)\ldots & & 34 & & 10 & & 2(14) & & 34 \\ & 1(15) & & 6 & & 1 & & 1(14) & & 6 & & 1 & & 1(14)\ldots & & 6 & & 1 & & 1(14) & & 6 & \end{pmatrix},$$

where the part $\begin{pmatrix} 34 & & 10 & & 2(14) & \\ & 1 & & 1(14) & & 6 \end{pmatrix}$ occurs $k-1$ times. Denote the upper left-hand i-rowed minor of its determinant by A_i. Then A_{16k+1} is the determinant of f_{16k+1}.

For $k=1$, the form in 17 variables

$$f_{17}=\begin{pmatrix} 2(15) & & 2 & & 34 \\ & 1(15) & & 6 & \end{pmatrix}$$

is non-decomposable by lemma 3. By lemma 1, $A_{15}=16$, $A_{16}=17$, and $A_{17}=34.17-6^2.16=2$.

Suppose now that for $k=m$, in (14) the form f_{16m+1} is non-decomposable and $A_{16m}=17$, $A_{16m+1}=2$. Take $k=m+1$, Then $A_{16m+2}=10.2-17=3$, $A_{16m+3}=2.3-2=4$, and so step by step, $A_{16m+16}=17$. $A_{16m+17}=34.17-6^2.16=2$.

From lemma 8, on taking

$$g_1=f_{16m+1},\ \mathfrak{D}_1=2,\ A=17,$$

$$g_2=\begin{pmatrix} 2(13) & & 2 & & 34 \\ & 1(13) & & 6 & \end{pmatrix},\quad \mathfrak{D}_2=34.15-6^2.14=6,$$

$$g_3=\begin{pmatrix} 2(14) & & 2 & & 34 \\ & 1(14) & & 6 & \end{pmatrix},\quad \mathfrak{D}_3=34.16-6^2.15=4,$$

and $a=8$, then $g=f_{16m+17}$ is non-decomposable, since from lemma 3, g_2, g_3 are non-decomposable. Hence f_{16k+1} is non-decomposable for any $k>0$.

Consider next the form in $22h+1$ variables

$$(15)\quad f'_{22h+1}=\begin{pmatrix} 2(21) & & 2 & & 24 & & 13 & & 2(20) & & 24 & & 13 & & 2(20)\ldots & & 24 & & 13 & & 2(20) & & 24 \\ & 1(21) & & 5 & & 1 & & 1(20) & & 5 & & 1 & & 1(20)\ldots & & 5 & & 1 & & 1(20) & & 5 & \end{pmatrix}$$

the part $\begin{pmatrix} 24 & & 13 & & 2(20) & \\ & 1 & & 1(20) & & 5 \end{pmatrix}$ occurring $h-1$ times. Denote the minors corresponding to the A's above by A'_i.

For $h=1$, the form in 23 variables

$$f'_{23}=\begin{pmatrix} 2_{(21)} & & 2 & & 24 \\ & 1_{(21)} & & 5 & \end{pmatrix}$$

is non-decomposable by lemma 3. By lemma 1, $A_{22}'=23$, and $A_{23}'=24.23-5^2.22=2$. Suppose now for $h=m$, in (15) the form f'_{22m+1} is non decomposable and that $A'_{22m+1}=23$, $A'_{22m+1}=2$. Take $h=m+1$. Then $A'_{22m+2}=13.2-23=3$, $A'_{22m+3}=2.3-2=4$, and so step by step, $A'_{22m+22}=23$, $A'_{22m+23}=24.23-5^2.22=2$.

From lemma 8, on taking

$$g_1=f'_{22m+1},\ \mathfrak{D}_1=2,\ A=23,$$

$$g_2=\begin{pmatrix} 2_{(19)} & & 2 & & 24 \\ & 1_{(19)} & & 5 & \end{pmatrix},\ \mathfrak{D}_2=24.21-5^2.20=4,$$

$$g_3=\begin{pmatrix} 2_{(20)} & & 2 & & 24 \\ & 1_{(20)} & & 5 & \end{pmatrix},\ \mathfrak{D}_2=24.22-5^2.21=3,$$

and $a=11$, then $g=f'_{22m+23}$ is non-decomposable, since by lemma 3, g_2, g_3 are non-decomposable. Hence f'_{22h+1} is non- decomposable for any $h>0$.

Finally, we consider the form in $16k+22h+1$ variables $f''_{16k+22h+1}=$

$$\begin{pmatrix} 2_{(15)} & & 2 & & 34 & & 10 & & 2_{(14)} & \dots & 34 & & 10 & & 2_{(14)} & & 34 & & 10 & & 2_{(20)} & & 24 & & 13 & & 2_{(20)} & \dots & 24 & & 13 & & 2_{(20)} & & 24 \\ & 1_{(15)} & & 6 & & 1 & & 1_{(14)} & & \dots & & 6 & & 1 & & 1_{(14)} & & 6 & & 1 & & 1_{(20)} & & 5 & & 1 & & 1_{(20)} & \dots & & 5 & & 1 & & 1_{(20)} & & 5 & \end{pmatrix}$$

with $k>0$, $h>0$. Denote the corresponding minors now by A''_i. Then

$$A''_{16k+1}=A_{16k+1}=2,\ A''_{16k+2}=A_{16k+2}=10.2-17=3,$$
$$A''_{16k+3}=A_3'=4,\ \text{etc.},\ A''_{16k+22h}=A'_{22h}=23,\ A''_{16k+22h+1}=A'_{22h+1}=2.$$

From lemma 8, on taking

$$g_1=f_{16k+1},\quad \mathfrak{D}_1=2,\ A=17,$$

$$g_2=\begin{pmatrix} 2_{(19)} & & 2 & & 24 \\ & 1_{(19)} & & 5 & \end{pmatrix},\quad \mathfrak{D}_2=4,$$

$$g_3=\begin{pmatrix} 2_{(20)} & & 2 & & 24 \\ & 1_{(20)} & & & \end{pmatrix},\quad \mathfrak{D}_3=3,\ a=8,$$

the form $g=f''_{16k+22h+1}$ is non-decomposable. Then as in the proof of the non-decomposability of f'_{22h+1}, we can show that $f''_{16k+22h+1}$ is non-decomposable for any $k>0$, $h>0$.

Now every integer $n=2m+1>176$ is of the form $16k+22h+1$, since $m=8k+11h$ has a solution with $h\geqq 0$, $k\geqq 0$ for $m>87$. Our lemma is proved.

LEMMA 11. *There exist even and odd non-decomposable forms in less than 13k variables with determinant $k+2$.*

Let r be an integer such that

$$(16)\qquad 10k > r^2 > 2k+4 \qquad (k>0).$$

Such integers always exist, for if we write

$$r^2 > 2k+4 \geqq (r-1)^2,$$

$$r^2 \leqslant (\sqrt{2k+4}+1)^2 = 2k+5+2\sqrt{2k+4}.$$

Then (16) holds, if

$$10k > 2k+5+2\sqrt{2k+4}$$

or

$$8(8k-11)k+9>0,$$

which is true for all $k>1$. If $k=1$, $r=3$ suffices.

Consider the form in r^2-k-2 variables

$$f_{r^2-k-2} = \begin{pmatrix} 2_{(r^2-k-4)} & 2 & r^2-1 \\ & 1_{(r^2-k-4)} & r \end{pmatrix}.$$

By lemma 1, its determinant is $(r^2-1)(r^2-k-2)-r^2(r^2-k-3)=k+2$. It is non-decomposable; for by lemma 3, it tuffices to show that

$$r^2-k-2>k+2, \qquad 2r\leqslant r^2-k-2.$$

The first inequality follows from (16). The second is true for $k=1$. For $k=2$, we can take $r=4$. For $k>2$, we have $r\geqq 4$. Suppose then the second ineqaulity is not true, i. e. $2r>r^2-k-2$, and so

$$(r-1)^2\leqslant k+2.$$

Then from $r^2\geqq 2k+5$, we get

$$2(r-1)^2<r^2,$$

which is false for $r\geqq 4$. Hence f_{r^2-k-2} is non-decomposable.

Consider next the form in $(r+1)^2-k-2$ variables with determinant $k+2$

$$f_{(r+1)^2-k-2} = \begin{pmatrix} 2_{((r+1)^2-k-4)} & 2 & (r+1)^2-1 \\ & 1_{((r+1)^2-k-4)} & r+1 \end{pmatrix}.$$

It is non-decomposable; for by lemma 3, it suffices to show that

$$(r+1)^2-k-2>k+2, \quad 2(r+1)\leqslant (r+1)^2-k-2.$$

Both of the inequalities follow from $r^2>2k+4$.

Since $(r+1)^2 - k - 2 < 13\,k$ and the number of variables of one of the forms $f_{r^2-k-2}, f_{(r+1)^2-k-2}$ is even and of the other is odd, the lemma is proved.

2. *Proofs of the theorems* 1, 2 *and* 3.

Theorem 1 evidently follows from lemma 5 and 9.

To prove theorem 2, we put $n = m + 1 + s$, where $s > 176$ is an odd integer and with the r of (16), $m = r^2 - k - 2$ or $(r+1)^2 - k - 2$, the choice being determined by $m \equiv n \pmod 2$. Let the form in s variables obtained in lemma 12 be f_s. Then the upper left-hand minor A_{s-1} is odd and > 1. Let

$$u = \tfrac{1}{2}(A_{s-1} + 3).$$

Then u is an integer and $0 < u - 2 < \frac{1}{2} A_{s-1}$. Suppose first $m = r^2 - k - 2$. Consider the form

$$f_n = f_s + 2x_s x_{s+1} + u x_{s+1}^2 + 2x_{s+1}\, x_{s+2} + f_{r^2-k-2}(x_{s+2}, \ldots, x_{s+r^2-k-1}),$$

where f_{r^2-k-2} is the form obtained from lemma 11. Denote the upper left-hand i-rowed principal minor of f_n by A_i. Then

$$A_s = 2, A_{s+1} = 2u - A_{s-1} = 3, \quad A_{s+2} = 2.3 - 2 = 4, \text{ etc.},$$
$$A_{s+r^2-k-3} = r^2 - k - 1\,, A_{s+r^2-k-2} = r^2 - k;$$

and so the determinant of f_n is $(r^2-1)(r^2-k) - r^2(r^2-k-1) = k$.

From lemma 8, on taking

$$g_1 = f_s, \mathfrak{D}_1 = 2, A = A_{s-1} \geqq 3, g_2 = f_{r^2-k-2}, \mathfrak{D}_2 = k+2,$$
$$g_3 = \begin{pmatrix} 2_{(r^2-k-3)} & & 2 & r^2-1 \\ & 1_{(r^2-k-3)} & & r \end{pmatrix}, \quad \mathfrak{D}_3 = k+1, a = u-2,$$

$g = f_n$ is non-decomposable, if g_3 is non-decomposable. By lemma 3, g_3 is non-decomposable, if $\mathfrak{D}_3 < r^2 - k - 1$, or $2k + 2 < r^2$ and this follows from the choice of r in lemma 11.

Similarly, f_n is non-decomposable if $m = (r+1)^2 - k - 2$.

Hence theorem 2 is proved.

To prove theorem 3, by theorem 1, we need only supply special results for $n = 6, 7, 9, 10, 11, 13, 17, 19, 23$.

Since $6 \equiv -2$, $7 \equiv -1$, $23 \equiv -1 \pmod 8$, by lemma 7, we have a non-decomposable form for $n = 6, 7$ [11]), and 23. For $n = 9, 10, 11, 13, 17, 19$, we have that by lemma 3 the forms

$$\begin{pmatrix} 15 & & 2_{(i)} & \\ & 4 & & 1_{(i)} \end{pmatrix} \quad (i+1 = 9, 10, 11, 13);$$

$$\begin{pmatrix} 24 & & 2_{(i)} & \\ & 5 & & 1_{(i)} \end{pmatrix} \quad (i+1 = 17, 19)$$

are non-decomposable.

In closing, we should like to thank Prof. Mordell for suggesting shorter proofs of lemmas 2, 3 and for his kind help with the manuscript.

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11) These are the same forms given by Prof. Mordell. See footnote [3]).