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# ON INTERPOLATION I

## QUADRATURE- AND MEAN-CONVERGENCE IN THE LAGRANGE-INTERPOLATION[1]

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Let

$$B \equiv \left\{ \begin{matrix} x_1^{(1)} & & & \\ x_1^{(2)}, & x_2^{(2)} & & \\ \vdots & & & \\ x_1^{(n)}, & x_2^{(n)}, & \cdots & x_n^{(n)} \\ \vdots & \vdots & & \vdots \end{matrix} \right\} \tag{1}$$

be an aggregate of points, where for every $n$

$$1 \geqq x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} \geqq -1. \tag{2}$$

Let $f(x)$ be defined in the interval $[-1, +1]$. We define the $n^{\text{th}}$ Lagrange-parabola of $f(x)$ with respect to $B$, as the polynomial of degree $\leqq n-1$, which takes at the points $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$ the values $f(x_1^{(n)}), f(x_2^{(n)}), \cdots, f(x_n^{(n)})$. We denote this polynomial by $L_n(f)$ and we sometimes omit to indicate its dependence upon $x$ and $B$. It is known,[2] that

$$L_n(f) = \sum_{i=1}^{n} f(x_i^{(n)})\, l_i^{(n)}(x) \equiv \sum_{i=1}^{n} f(x_i)\, l_i(x). \tag{3}$$

The functions $l_i(x)$, called the fundamental functions of the interpolation, are polynomials of degree $n-1$ and if

$$\omega(x) \equiv \omega_n(x) = \prod_{i=1}^{n} (x - x_i) \tag{4}$$

then

$$l_i(x) = \frac{\omega(x)}{\omega'(x_i)(x - x_i)}. \tag{5}$$

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[1] This paper was partly read at the Math. and Phys. Association, Budapest, May 26, 1934.

[2] In the symbol $l_i^{(n)}(x)$ the letter $n$ is an index, and does not indicate the $n^{\text{th}}$ differential quotient. In the paragraphs 1. and 2. as far as possible we shall not explicitly denote the dependence upon $n$.

It is known that if $\psi(x)$ is a polynomial of the $m^{\text{th}}$ degree, then

$$L_{m+k}(\psi) \equiv \psi(x) \qquad k = 1, 2, \cdots. \tag{6}$$

When $\psi(x) \equiv 1$, we obtain from (6) and (3)

$$\sum_{i=1}^{n} l_i(x) \equiv 1. \tag{6a}$$

The first problem is that of convergence, i.e. if $f(x)$ and $B$ are given, we ask whether or not at any given $x_0$ the sequence of polynomials $L_n(f)$ tends to $f(x_0)$. Suppose $f(x)$ to be a continuous function; then according to the well known theorem of Faber,[3] to any given $B$ we may find a *continuous* function[4] $f_1(x)$ such that in the interval $[-1, +1]$ the $L_n(f_1)$ parabolas do not converge uniformly to $f_1(x)$. In 1931 Bernstein[5] proved that, given any $B$, we may find a continuous function $f_2(x)$ such that the sequence $L_n(f_2)$ is *unbounded* at a certain fixed $\xi_0$ where $-1 \leqq \xi_0 \leqq +1$. The proof is based upon the following theorem of Hahn:[6]

The necessary and sufficient condition that

$$\lim_{n\to\infty} L_n(f)_{x=x_0} = f(x_0)$$

for a given $B$, at a given $x_0(-1 \leqq x_0 \leqq +1)$, for *any* continuous $f(x)$, is that

$$A(n) \equiv \sum_{\nu=1}^{n} | l_\nu(x_0) | < C \tag{7}$$

where $C$ is a positive constant independent of $n$. Thus it was only necessary to prove that for any given $B$ we could find a $\xi_0$ in $[-1, +1]$ such that the sequence (7) was unbounded, if $n \to \infty$.

If we are to prove the divergence not at a certain $x_0$, but at a countable aggregate in the interval $[-1, +1]$, we obtain a sufficient condition in the following generalisation of Hahn's theorem.[7] Let

$$A(x, n) = \sum_{\nu=1}^{n} | l_\nu(x) | \tag{8}$$

be an unbounded sequence of numbers for *any* fixed $x(-1 \leqq x \leqq +1)$. In this case for *any* countable aggregate $\Omega_1$ in $[-1, +1]$ we can find a continuous

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[3] G. Faber: *Über die interpolatorische Darstellung stetiger Funktionen.* Jahresbericht der Deutschen Mathematikervereinigung, Bd. 23. 1914, S. 190–210.

[4] In the whole of this paper the expression "continuous" denotes a function continuous in the whole of the interval $[-1, +1]$.

[5] Bernstein: *Sur la limitation des valeurs* etc. Bull. Acad. Sc. de l'URSS. 1931. No. 8. 1025–1050.

[6] H. Hahn: über das *Interpolationsproblem.* Math. Zeitschrift 1, 1918. 115–142. His proof is based on a general principle of Lebesgue.

[7] Banach-Steinhaus: Sur le principe de la condensation des ungularités Fundamenta Math. (1927).

$f_3(x)$ so, that $L_n(f_3)$ is unbounded for any element of $\Omega_1$. G. Grünwald[8] proved that (8) holds for an important class of matrices, the $n^{\text{th}}$ line of which is provided by the real roots of Jacobi's polynomial $J_n(x, \alpha, \beta)$. To put it roughly the Lagrange interpolation taken for any $B$ is "bad" from the point of view of convergence.

Let now $B$ be defined so that its $n^{\text{th}}$ line is given by the $n$ different real roots in $(-1, +1)$ of the Tschebyscheff-polynomial $T_n(x)$ (for which $T_n(\cos\vartheta) = \cos n\vartheta$). Then we may easily verify that[9]

$$(9)\quad \begin{cases} L_n(f) = \alpha_0 + \sum_{r=1}^{n-1} \alpha_r \cos r\vartheta \\ \text{where} \\ \alpha_0 = \frac{1}{n}\sum_{k=1}^{n} f(x_k), \qquad \alpha_r = \frac{2}{n}\sum_{k=1}^{n} f(x_k)\cos r\,\frac{2k-1}{2n}\pi, \\ x_k = \cos\frac{2k-1}{2n}\pi \qquad k = 1, 2, \cdots n. \\ \qquad\qquad\qquad\qquad r = 1, 2, \cdots n-1. \end{cases}$$

By a heuristic limiting process we might obtain the Fourier series of $f(\cos\vartheta)$, which indicates an interesting analogy between these special interpolation parabolas and the Fourier series of $f(\cos\vartheta)$. This analogy also appears in many other relations e.g. the form of (8) taken for $T$, which determines the convergence, is completely analogous to Lebesgue's constants, well known in the theory of Fourier-series. Another analogy: We know that for any given countable aggregate $\Omega_2$ we may find a continuous $f(x)$ so that the partial sums of its Fourier-series are uniformly bounded in $[0, 2\pi]$ and nevertheless they oscillate at every point of $\Omega_2$. For the Lagrange interpolation we proved that for any given countable $\Omega_3$ we may find a continuous $f_4(x)$ so that its Lagrange-parabolas with respect to $T$ are uniformly bounded in $[-1, +1]$ and nevertheless they do not converge to $f_4(x)$ at the points of $\Omega_3$. For the present we omit the proof. We only indicate that it is based upon the well known construction-principle of Lebesgue. Connected with these facts and others that we did not mention, the following result is very astonishing. There is a continuous $f_5(x)$ such that the first-order arithmetical means of its Lagrange parabolas at $x = 0$ taken for $T$

$$\phi_n(0) = \frac{L_1(f_5)_0 + \cdots + L_n(f_5)_0}{n}$$

are unbounded. We omit the proof; we may obtain it without any difficulty from Hahn's theorem. It seems that the same is true for the arithmetical means of any order; as far as we know this is not decided as yet. We do not even know with certainty whether or not the convergent Lagrange-parabolas of a

[8] Oral communication.

[9] L. Fejér, *Die Abschätzung eines Polynoms* etc. Math. Zeitschrift 32, 1930, 426.

continuous function taken for any $B$ at $x_0$ can converge to another value than $f(x_0)$. The remarks above show that the question is difficult. Nevertheless a certain analogy between the Fourier-series and these special interpolation parabolas for the arithmetical means may be preserved in a way found by Prof. Fejér.[10]

The second group of questions is concerned with the so called "quadrature-convergence" i.e. the examination of the sequence $\int_{-1}^{1} L_n(f)dx$, $(n = 1, 2, \cdots)$. Stieltjes[11] proved that if the $n^{\text{th}}$ line of $B$ is given by the $n$ different real roots of $P_n(x) = 0$, $P_n(x)$ being the $n^{\text{th}}$ Legendre-polynomial, then the integrals of the Lagrange parabolas belonging to any bounded and $R$ integrable ("$R$ integrable" means a function integrable in Riemann's sense in $[-1, +1]$) $f(x)$ tend to $\int_{-1}^{1} f(x)dx$, or as we shall say in the following pages: there is quadrature-convergence for this matrix. Since then Fejér[12] and Szegö[13] gave a powerful generalisation of this theorem. Pólya[14] proved that we have quadrature-convergence for continuous $f(x)$ if and only if the sequence

$$A_1(n) = \sum_{k=1}^{n} |\lambda_k^{(n)}|$$

remains below a bound independent of $n$; here $\lambda_k^{(n)} = \int_{-1}^{1} l_k^{(n)}(x)dx$, $n = 1, 2, \cdots$, $k = 1, 2, \cdots n$. $\lambda_k^{(n)}$ are the so called "Cotes numbers" of the matrix. Fejér[12] proved that the positiveness of all of the $\lambda_k^{(n)}$ is sufficient for the quadrature-convergence. In the following pages we examine instead of the quadrature-convergence the so called mean convergence, which requires more than the previous one. Mean convergence requires for *any* bounded and $R$ integrable function $f(x)$

$$\lim_{n\to\infty} \int_{-1}^{1} [f(x) - L_n(f)]^2\, dx = 0. \tag{10}$$

In §1 we show very simply that for a general class of matrices there is mean convergence.

The matrix-class in question is given in the following theorem.

THEOREM I. *Let $p(x)$ be a function such that*

$$p(x) \geqq M > 0 \qquad -1 \leqq x \leqq +1 \tag{11a}$$

[10] L. Fejér: *Über Interpolation.* Göttinger Nachrichten 1916. 66–91.

[11] Stieltjes: Oeuvres Bd. I. 377–395.

[12] L. Fejér, *Mechanische Quadraturen mit positiven Cotesschen Zahlen.* Math. Zeitschr. 37, 1933, 287–310.

[13] G. Szegö, *Asymptotische Entwicklungen der Jacobischen Polynome,* Schriften der Königsberger Gelehrten Gesellschaft, 1933.

[14] G. Pólya: *Über die Konvergenz von Quadraturverfahren,* Math. Zeitschr. 37. 1933. 264–287.

(11b) $$\int_{-1}^{1} p(x)\,dx \quad \textit{exists.}$$

*It is known that there is an infinite sequence of polynomials* $\omega_0(x), \omega_1(x), \cdots$ *where the degree of* $\omega_n(x)$ *is n with*

$$\int_{-1}^{1} \omega_n(x)\omega_m(x)p(x)\,dx \begin{matrix} \neq 0 \\ = 0 \end{matrix} \text{ if } \begin{matrix} n = m \\ n \neq m \end{matrix}; \textit{ coefficient of } x^n \textit{ in } \omega_n(x) = 1.$$

*As known* $\omega_n(x)$ *has in* $[-1, +1]n$ *different real roots. Then our relation* (10) *is true for any matrix formed of these roots.* Or more generally,

THEOREM Ia. *Let* $\omega_n(x)$ *be the above polynomials,* $A_n$ *and* $B_n$ *constants such that the equation*

(12) $$R_n(x) \equiv x^n + \cdots \equiv \omega_n(x) + A_n\omega_{n-1}(x) + B_n\omega_{n-2}(x) = 0$$

*may have in* $[-1, +1]n$ *different real roots and* $B_n \leqq 0$*; then (10) holds also for the matrices formed by these roots.*

In §1 we prove this Theorem Ia. We communicated our Theorem I to Professor Szegö and he found for it essentially the same proof as we did.

The restriction on the roots of (12) is not very great, for it is evident that in $[-1, +1]$ there are always at least $(n - 2)$ changes of sign.

We prove Theorem Ia by proving the relation

(13) $$\lim_{n\to\infty} \int_{-1}^{1} [f(x) - L_n(f)]^2 p(x)\,dx = 0,$$

which will be shown to be a consequence of $p(x) \geqq M$ and of the existence of $\int_{-1}^{1} p(x)\,dx$. From (13) it follows by (11a) that

$$0 \leqq \int_{-1}^{1} [f(x) - L_n(f)]^2\,dx \leqq \frac{1}{M} \int_{-1}^{1} [f(x) - L_n(f)]^2 p(x)\,dx,$$

and this by (13) establishes Theorem Ia.

COROLLARY OF THEOREM Ia. *For all bounded and R integrable* $f(x)$ *we have for the matrices given in Theorem Ia*

(14) $$\lim \int_{-1}^{1} |f(x) - L_n(f)|\,dx = 0$$

*and a fortiori there is quadrature convergence for these matrices.*

Considering only the quadrature convergence or rather the validity of the more rigorous (14) we shall prove the following more precise

THEOREM II. *Let*

(15a) $$p(x) \geqq 0\ [-1, +1];$$

*suppose further the existence of*

(15b) $$\int_{-1}^{1} p(x)dx \textit{ and } \int_{-1}^{1} \frac{1}{p(x)}\,dx.$$

*If the polynomials* (12) *are formed with the orthogonalised polynomials* $\omega_n(x)$ *belonging to such* $p(x)$, (14) *holds for the corresponding matrix.*

In Theorem Ia and II we have—as far as we know—the first general theorem for mean and quadrature-convergence.

Now we mention some interesting special matrix-classes. Let

$$p(x) = (1 - x)^\alpha (1 + x)^\beta \qquad \begin{matrix} -1 < \alpha \leqq 0 \\ -1 < \beta \leqq 0 \end{matrix}$$

and $A_n = B_n = 0$. Then evidently (12) is satisfied i.e. there is mean convergence. The $\omega_n(x)$ belonging to this moment-function are the Jacobi-polynomials for the parameters $\alpha$ and $\beta$. If $\alpha = \beta = 0$, we have the case of the Legendre-polynomials $P_n(x)$; for $\alpha = \beta = -\frac{1}{2}$ the Tschebischeff polynomials $T_n(x)$.

Let now $\alpha = \beta = 0$, and $A_n$ and $B_n$ such that (12) be satisfied. Fejér[12] proved the quadrature-convergence for this matrix class; as we see, here we obtained mean convergence. If $\alpha = \beta = 0, A_n = 0, B_n = -1$ we have $R_n(x) \equiv P_n(x) - P_{n-2}(x)$ and obtain a matrix conspicuous by its interesting extremal properties.[15] Now suppose

$$p(x) = (1 - x)^\alpha (1 + x)^\beta \qquad \begin{matrix} -1 < \alpha < 1 \\ -1 < \beta < 1 \end{matrix}$$

and $A_n = B_n = 0$ and consider only (14), which—as Theorem II shows—is satisfied. Szegö[13] proved that for max $(\alpha, \beta) < \frac{3}{2}$ we have quadrature-convergence but for max $(\alpha, \beta) > \frac{3}{2}$ we have not. Thus, by the special case of our Theorem II, we obtain a new proof of Szegö's theorem for $-1 < \alpha < 1$, $-1 < \beta < 1$; we even obtain more, for according to Szegö $\int_{-1}^{1} (f(x) - L_n(f))dx \to 0$ whereas we have $\int_{-1}^{1} | f(x) - L_n(f) | \, dx \to 0$, as $n \to \infty$. For $\alpha = \beta = \frac{1}{2}$ we have the case of the Tschebischeff-polynomials $U_n(x)$, where $U_n(\cos \vartheta) = \sin(n + 1)\vartheta/\sin \vartheta$. If we consider the interpolating parabolas, instead of in $[-1, +1]$, only in $[-1 + \epsilon, 1 - \epsilon]$, then, as here $(1 - x)^\alpha (1 + x)^\beta \geqq M > 0$, we obtain for *all* $\alpha, \beta > -1$

$$\lim_{n\to\infty} \int_{-1+\epsilon}^{1-\epsilon} [f(x) - L_n(f)]^2 \, dx = 0.$$

This is interesting from the point of view of Szegö's result. It shows that in the case of max $(\alpha, \beta) > \frac{3}{2}$ the divergence is due to the rapid growth of the parabolas on the margin.

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[15] L. Fejér, *Bestimmung derjenigen Abscissen* etc., Annali della R. Scuola Normale Superiore di Pisa, serie II, Vol. I, 1932.

We immediately obtain Theorem II from (13), viz.

$$0 \leqq \int_{-1}^{1} | f(x) - L_n(f) | \, dx = \int_{-1}^{1} | f(x) - L_n(f) | \sqrt{p(x)} \, \frac{dx}{\sqrt{p(x)}}$$

$$\leqq \sqrt{\int_{-1}^{1} [f(x) - L_n(f)]^2 \, p(x) \, dx \cdot \int_{-1}^{1} \frac{dx}{p(x)}}$$

and by (13) this proves (14).

From the above mentioned theorem of Szegö we see that there are $B$ matrices, for which we have no quadrature-convergence and thus a fortiori no mean convergence. The most important problem in this connection would be to give the necessary and sufficient condition of the mean convergence. Our Theorem III gives a necessary condition for the mean convergence. It asserts

THEOREM III. *If the sequence*

$$C(n) = \sum_{k=1}^{n} \int_{-1}^{1} l_k(x)^2 \, dx$$

*is unbounded as* $n \to \infty$, *there exists a continuous* $f_6(x)$ *such that for our matrix*

$$\overline{\lim_{n \to \infty}} \int_{-1}^{1} [f_6(x) - L_n(f_6)]^2 \, dx = +\infty.$$

## §1

As explained in the introduction we have to prove (13) for the fundamental points given by the roots of the $R_n(x)$ polynomials of (12). First we prove[16] that

$$\text{(16a)} \qquad \int_{-1}^{1} l_i(x) \, p(x) dx \geqq 0 \qquad i = 1, 2, \cdots, n$$

and

$$\text{(16b)} \qquad \sum_{i=1}^{n} \int_{-1}^{1} l_i(x)^2 \, p(x) dx \leqq \int_{-1}^{1} p(x) dx.$$

Consider the expression

$$\int_{-1}^{1} [l_i(x)^2 - l_i(x)] \, p(x) dx.$$

But $l_i(x)^2 - l_i(x) = R_n(x)F(x)$, where $F(x)$ is a polynomial of degree $(n - 2)$, in which the coefficient of the highest term is evidently $1/\omega_n'(x_i)^2$. Thus if $F(x) = c_0\omega_0(x) + \cdots + \omega_{n-2}(x)/\omega_n'(x_i)^2$, by the orthogonality of the $\omega_n(x)$'s we have

$$\int_{-1}^{1} [l_i(x)^2 - l_i(x)] \, p(x) dx = \frac{B_n}{\omega_n'(x_i)^2} \int_{-1}^{1} \omega_{n-2}(x)^2 \, p(x) dx \leqq 0$$

[16] For $p(x) \equiv 1$ the proof of (16a) is to be found in Fejér's[12] paper.

i.e.

$$\text{(16c)} \qquad \int_{-1}^{1} l_i(x)^2\, p(x)dx \leqq \int_{-1}^{1} l_i(x)\, p(x)dx$$

which immediately establishes (16a); by summation for $i = 1, 2, \cdots, n$ we obtain (16b) in consequence of (6a).

Let now $\Omega_4$ be an aggregate in $[-1, +1]$ formed of closed non-overlapping intervals. We prove that

$$\text{(17)} \qquad \sum_{\substack{i \\ x_i^{(n)} \{\Omega_4}} \sum_{\substack{k \\ x_k^{(n)} \{\Omega_4}} \left| \int_{-1}^{1} l_i(x)\, l_k(x)\, p(x)dx \right| \leqq 2 \sum_{x_i^{(n)} \{\Omega_4} \int_{-1}^{1} l_i(x)^2\, p(x)dx.$$

First we assert that for every $k, i$ with $1 \leqq k \leqq n, 1 \leqq i \leqq n$

$$\text{(18)} \qquad (-1)^{i+k+1} I_{ik} \equiv (-1)^{i+k+1} \int_{-1}^{1} l_i(x)\, l_k(x)\, p(x)dx \geqq 0 \quad \text{if} \quad i \neq k.$$

For by (5)

$$I_{ik} = \frac{1}{R_n'(x_i)\, R_n'(x_k)} \int_{-1}^{1} \frac{R_n(x)}{(x - x_i)\,(x - x_k)} R_n(x)\, p(x)dx.$$

As $i \neq k$,

$$\frac{R_n(x)}{(x - x_k)\,(x - x_i)} = d_0\, \omega_0(x) + \cdots + d_{n-3}\, \omega_{n-3}(x) + \omega_{n-2}(x).$$

Hence considering the definition of $R_n(x)$ we have

$$I_{ik} = \frac{B_n}{R_n'(x_i)\, R_n'(x_k)} \int_{-1}^{1} \omega_{n-2}(x)^2\, p(x)dx$$

which proves (18), as $B_n \leqq 0$ and sign $R_n'(x_i)\, R_n'(x_k) = (-1)^{i+k}$. Thus we have

$$\sum_{\substack{i \\ x_i^{(n)} \{\Omega_4}} \sum_{\substack{k \\ x_k^{(n)} \{\Omega_4}} \left| \int_{-1}^{1} l_i(x)\, l_k(x)\, p(x)dx \right|$$

$$= \sum_{x_i^{(n)} \{\Omega_4} \int_{-1}^{1} l_i(x)^2\, p(x)dx - \sum_{x_i^{(n)} \{\Omega_4} \sum_{x_k^{(n)} \{\Omega_4} (-1)^{i+k} \int_{-1}^{1} l_i(x)\, l_k(x)\, p(x)dx$$

$$= 2 \sum_{x_i^{(n)} \{\Omega_4} \int_{-1}^{1} l_i(x)^2\, p(x)dx - \int_{-1}^{1} [\sum_{x_i^{(n)} \{\Omega_4} (-1)^i\, l_i(x)]^2\, p(x)dx$$

$$\leqq 2 \sum_{x_i^{(n)} \{\Omega_4} \int_{-1}^{1} l_i(x)^2\, p(x)dx;$$

thus (17) is proved.

If $\Omega_4$ denotes the whole of the interval $[-1, +1]$, in consequence of (16b) we have

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \left| \int_{-1}^{1} l_i(x)\, l_k(x)\, p(x)dx \right| \leqq 2 \int_{-1}^{1} p(x)dx. \tag{19}$$

Let now $f(x)$ be continuous, $\varphi(x)$ the polynomial of degree $n - 1$ that gives the best approximation to it in Tschebischeff's sense for the interval $[-1, +1]$. Write

$$f(x) - \varphi(x) = \Delta(x) \tag{20}$$

$$\max_{|x| \leqq 1} |f(x) - \varphi(x)| = E_{n-1} \tag{21}$$

$$I_n = \int_{-1}^{1} [f(x) - L_n(f)]^2\, p(x)dx.$$

Then by (6) we have

$$\begin{aligned} I_n = \int_{-1}^{1} [\Delta(x) - L_n(\Delta)]^2\, p(x)dx &\leqq 2 \int_{-1}^{1} \Delta(x)^2\, p(x)dx \\ &+ 2 \int_{-1}^{1} L_n(\Delta)^2\, p(x)dx \equiv I_n' + I_n''. \end{aligned} \tag{22}$$

We have evidently

$$I_n' \leqq 2\, E_{n-1}^2 \int_{-1}^{1} p(x)dx. \tag{23}$$

Further by (3)

$$\begin{aligned} |I_n''| &= 2 \left| \sum_{i=1}^{n} \sum_{k=1}^{n} \Delta(x_i)\, \Delta(x_k) \int_{-1}^{1} l_i(x)\, l_k(x)\, p(x)dx \right| \\ &\leqq 2\, E_{n-1}^2 \sum_{i=1}^{n} \sum_{k=1}^{n} \left| \int_{-1}^{1} l_i(x)\, l_k(x)\, p(x)dx \right| \leqq 4\, E_{n-1}^2 \int_{-1}^{1} p(x)dx \end{aligned} \tag{24}$$

by (19); from (22), (23) and (24) we have

$$|I_n| \leqq 6E_{n-1}^2 \int_{-1}^{1} p(x)dx \tag{25}$$

which by Weierstrass' theorem establishes (13) for any continuous $f(x)$.

Now we require a Lemma.

Fejér's[12] theorem asserts: if for any aggregate of points $B$ the "Cotes numbers" $\int_{-1}^{1} l_i(x)dx$ are non-negative for any $i$ and $n$, then we have quadrature convergence. It may be proved in the very same way that if for a given matrix the "Cotes numbers belonging to the non negative and $R$ integrable $p(x)$"

$$\int_{-1}^{1} l_i(x)p(x)dx \tag{26}$$

are non-negative for any $i$ and $n$ and $f(x)$ is bounded and $R$ integrable, then

$$\lim_{n\to\infty} \int_{-1}^{1} L_n(f)p(x)dx = \int_{-1}^{1} f(x)p(x)dx.$$

It would be superfluous to repeat the proof. We require this in the proof of our

LEMMA. *Let $B_1$ be a matrix satisfying* (26), *and $\Omega_5$ be a set of a finite number of non-overlapping intervals in $[-1, +1]$; then for $n > n_0$ we have*

$$\sum_{\substack{i \\ x_i^{(n)} \{\Omega_5}} \int_{-1}^{1} l_i(x)p(x)dx < 2 \int_{\Omega_5} p(x)dx. \tag{27}$$

PROOF. We easily obtain this result if we consider the function $\psi(x)$ having the value 1 for points of $\Omega_5$ and 0 elsewhere. $\psi(x)$ is evidently bounded and $R$ integrable, so that according to Fejér's theorem

$$\lim_{n\to\infty} \int_{-1}^{1} L_n(\psi)p(x)dx = \int_{-1}^{1} \psi(x)p(x)dx. \tag{28}$$

But by the definition of $\psi(x)$ we may write

$$\int_{-1}^{1} L_n(\psi)p(x)dx = \sum_{\substack{i \\ x_i^{(n)} \{\Omega_5}} \int_{-1}^{1} l_i(x)p(x)dx, \tag{29}$$

further

$$\int_{-1}^{1} \psi(x)p(x)dx = \int_{\Omega_5} p(x)dx. \tag{30}$$

(27) is an evident consequence of (28), (29) and (30).

Now we consider the matrix B defined as in Theorem Ia. In consequence of (16a) the Lemma is applicable; we obtain from (16c) and (27)

$$\sum_{\substack{i \\ x_i^{(n)} \{\Omega_5}} \int_{-1}^{1} l_i(x)^2 p(x)dx \leqq \sum_{\substack{i \\ x_i^{(n)} \{\Omega_5}} \int_{-1}^{1} l_i(x)p(x)dx < 2 \int_{\Omega_5} p(x)dx, \tag{31}$$

and finally from (17)

$$\sum_{\substack{i \\ x_i^{(n)} \{\Omega_5}} \sum_{\substack{k \\ x_k^{(n)} \{\Omega_5}} \left| \int_{-1}^{1} l_i(x)l_k(x)p(x)dx \right| < 4 \int_{\Omega_5} p(x)dx. \tag{32}$$

Let now $f(x)$ be any bounded and $R$ integrable function. Then in virtue of the Riemann integrability, to any $\epsilon$ we can find a finite aggregate of non-overlapping open intervals of total length $\leqq \epsilon$ such that if we exclude these intervals, the oscillation of the function is $\leqq \epsilon$ at any point of the remaining aggregate $\Omega_6$. We now define $f_7(x)$ as follows: 1. in $\Omega_6$ let $f_7(x) \equiv f(x)$. 2. if we denote the excluded intervals by $(p_1, q_1) \cdots (p_\nu, q_\nu)$, ($\nu$ finite), the function $f_7(x)$ is represented in $(p_i, q_i)$ by the straight line connecting the point $(p_i, f(p_i))$ and $(q_i, f(q_i))$. Thus we define $f_7(x)$ for the whole of $[-1, +1]$, and its oscillation is at any point $\leqq \epsilon$. But then $f_7(x)$ may be uniformly approximated by a polynomial $\varphi(x)$ to within $2\epsilon$. Let the degree of $\varphi(x)$ be $m = m(\epsilon)$. Then we have

$$I_n \equiv \int_{-1}^{1} [f(x) - L_n(f)]^2 p(x)dx \leqq 2 \int_{-1}^{1} [f_7(x) - L_n(f_7)]^2 p(x)dx$$

$$(33) \qquad + 2 \int_{-1}^{1} [f - f_7 - L_n(f - f_7)]^2 p(x)dx \leqq 2 \int_{-1}^{1} [f_7(x) - L_n(f_7)]^2 p(x)dx$$

$$+ 4 \int_{-1}^{1} [f - f_7]^2 p(x)dx + 4 \int_{-1}^{1} L_n(f - f_7)^2 p(x)dx \equiv J_n' + J_n'' + J_n''',$$

say. As the degree of approximation to $f_7(x)$ is $2\epsilon$, we have by (25) for $n > m(\epsilon)$

$$(34) \qquad |J_n'| \leqq 24\epsilon^2 \int_{1}^{1} p(x)dx.$$

Further as $f(x) - f_7(x)$ differs from 0 only upon intervals, of which the total length is $\leqq \epsilon$ and as $|f(x) - f_7(x)| \leqq 2 \max_{|x| \leqq 1} |f(x)| \equiv 2M$, we have

$$(35) \qquad |J_n''| \leqq 16\, M^2 \sum_{i=1}^{\nu} \int_{p_i}^{q_i} p(x)dx.$$

For $J_n'''$ we may evidently write

$$J_n''' = \sum_{i=1}^{n} \sum_{k=1}^{n} (f(x_i) - f_7(x_i))(f(x_k) - f_7(x_k)) \int_{-1}^{1} l_i(x) l_k(x) p(x)dx.$$

In consequence of the definition of $f_7(x)$ the terms of this sum differ from 0 only when $x_i$ and $x_k$ lie in intervals $(p_l, q_l)$ and $(p_\mu, q_\mu)$ respectively. Hence

$$(36) \qquad |J_n'''| \leqq 4M^2 \sum_{\substack{i \\ x_i \{(p_l,\, q_l) \\ l = 1, 2, \cdots \nu}} \sum_{\substack{k \\ x_k \{(p_\mu,\, q_\mu) \\ \mu = 1, 2, \cdots \nu}} \left| \int_{-1}^{1} l_i(x) l_k(x) p(x)dx \right| \leqq 16M^2 \sum_{i=1}^{\nu} \int_{p_i}^{q_i} p(x)dx$$

by (32).

As the total length of the range of integration is $\leqq \epsilon$, it is evident by (33), (34), (35) and (36), that $I_n \to 0$ as $n \to \infty$. Hence the result.

## §2

In this paragraph we shall prove Theorem III. Let us write

$$S(n) = \sum_{i=1}^{n} \int_{-1}^{1} l_i(x)^2 dx,$$

and suppose this to be unbounded as $n \to \infty$. We shall prove that we can find a continuous function $f(x)$ with

$$\limsup_{n\to\infty} \int_{-1}^{1} [f(x) - L_n(f)]^2 dx = +\infty.$$

By hypothesis there exists an infinite sequence $n_1 < n_2 < \cdots$ with $S(n_1) < S(n_2) < \cdots \to \infty$. For the sake of simplicity of notation we denote by $m$ the $m^{\text{th}}$ element of this sequence $n_m$.

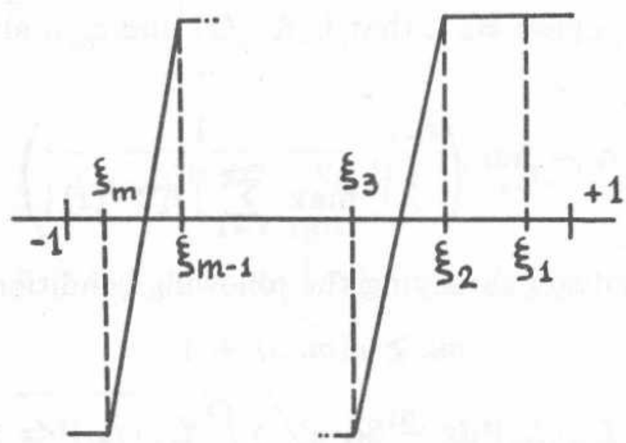


FIG. 1

Let the $m^{\text{th}}$ fundamental points be $1 \geqq \xi_1^{(m)} > \xi_2^{(m)} > \cdots > \xi_m^{(m)} \geqq -1$. We regard them as abscissas and to any $\xi_i^{(m)}$ we adjoin an ordinate $\epsilon_i$, where $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$ have arbitrarily the values $+1$ or $-1$. Thus we have $m$ points; we connect them as in Fig. 1 and obtain a continuous function $\psi_\epsilon(x)$ with

(37a) $$|\psi_\epsilon(x)| \leqq 1 \text{ for } -1 \leqq x \leqq +1$$

and

(37b) $$\int_{-1}^{1} L_m(\psi_\epsilon)^2 dx = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \epsilon_\mu \epsilon_\nu \int_{-1}^{1} l_i(x) l_k(x) dx.$$

By variation of the $\epsilon$'s we obtain $2^m$ different $\psi_\epsilon(x)$ functions. For these functions we have by forming the sums of (37b)

(38) $$\frac{1}{2^m} \sum_\epsilon \int_{-1}^{1} L_m(\psi_\epsilon)^2 dx = \sum_{\nu=1}^{m} \int_{-1}^{1} l_\nu(x)^2 dx = S(m),$$

hence we may choose $\epsilon$'s, so that for the corresponding $\psi_\epsilon(x)$ which we simply denote by $\psi(x)$, we have

$$\int_{-1}^{1} L_m(\psi)^2\,dx \geqq S(m). \tag{39}$$

According to Weierstrass, $\psi(x)$ may be approximated by a polynomial $f_m(x)$ of degree $\mu(m)$ so that

$$|f_m(x)| \leqq \tfrac{3}{2} \qquad -1 \leqq x \leqq +1 \tag{40a}$$

and

$$\int_{-1}^{1} L_m(f_m)^2\,dx \geqq \tfrac{1}{2}S(m). \tag{40b}$$

Now we select a partial sequence $f_{m_1}, f_{m_2}, \cdots$ of the sequence $f_1(x), f_2(x), \cdots$ and define a sequence of constants $c_1, c_2, \cdots$ in the following way. Let $f_{m_1}(x) = f_1(x)$ and $c_1 = 1$. Suppose $m_{r-1}$, that is $f_{m_{r-1}}(x)$ and $c_{r-1}$, already defined, then we define

$$c_r = \min\left(\frac{c_{r-1}}{4}, \frac{1}{\max\limits_{|x|\leqq 1} \sum\limits_{k=1}^{m_{r-1}} |l_k^{(m_{r-1})}(x)|}\right) \tag{41}$$

and $m_r$ as the least integer satisfying the following conditions:

$$m_r \geqq \mu(m_{r-1}) + 1 \tag{42$\alpha$}$$

$$c_r^2 \int_{-1}^{1} L_{m_r}(f_{m_r})^2\,dx - 8c_r \sqrt{2\int_{-1}^{1} L_{m_r}(f_{m_r})^2\,dx} > 4^r; \tag{42$\beta$}$$

these 2 conditions can evidently be satisfied in consequence of (40b) and $\lim_{m\to\infty} S(m) = \infty$.

We now form with these $c_r$ and $f_{m_r}(x)$ the function

$$f(x) = \sum_{r=1}^{\infty} c_r f_{m_r}(x). \tag{43}$$

We shall prove that this is the function postulated in our Theorem III.

By (41)

$$c_r \leqq \frac{1}{4^r} \tag{44}$$

and in consequence of (44) and (40a) it is evident that the infinite series for $f(x)$ uniformly converges in $[-1, +1]$ i.e. $f(x)$ is continuous.

Now we consider $L_{m_r}(f)$ for a fixed value $\rho$ of $r$. According to (42$\alpha$)

$$L_{m_\rho}(f) = \sum_{r=1}^{\rho-1} c_r f_{m_r}(x) + \sum_{r=\rho}^{\infty} c_r L_{m_\rho}(f_{m_r});$$

hence

$$(45)\quad \begin{aligned} I_{m_\rho} &= \int_{-1}^{1} [L_{m_\rho}(f) - f]^2 dx \\ &= \int_{-1}^{1} \left[ c_\rho\, L_{m_\rho}(f_{m_\rho}) + \sum_{r=\rho+1}^{\infty} c_r\, L_{m_\rho}(f_{m_r}) - \sum_{r=\rho}^{\infty} c_r f_{m_r}(x) \right]^2 dx. \end{aligned}$$

But in consequence of (44)

$$(46a)\quad \left| \sum_{r=\rho}^{\infty} c_r\, f_{m_r}(x) \right| \leq \tfrac{3}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right) = 2,$$

and in accordance with (40a) and (41)

$$(46b)\quad \sum_{r=\rho+1}^{\infty} c_r\, L_{m_\rho}(f_{m_r}) \leq \sum_{r=\rho+1}^{\infty} c_r \cdot \tfrac{3}{2} \sum_{\nu=1}^{m_\rho} | l_\nu^{(m_\rho)}(x) | \leq \tfrac{3}{2}(1 + \tfrac{1}{4} + \cdots) = 2.$$

From (45), (46a) and (46b)

$$(47)\quad I_{m_\rho} = \int_{-1}^{1} [c_\rho\, L_{m_\rho}(f_{m_\rho}) - 4\,\theta]^2\, dx \qquad \text{with } |\,\theta\,| \leq 1.$$

Further

$$\begin{aligned} I_{m_\rho} &> c_\rho^2 \int_{-1}^{1} L_{m_\rho}(f_{m_\rho})^2\, dx - 8c_\rho \int_{-1}^{1} |\, L_{m_\rho}(f_{m_\rho})\,|\, dx - 16 \\ &> c_\rho^2 \int_{-1}^{1} L_{m_\rho}(f_{m_\rho})^2\, dx - 8c_\rho \left[ 2 \int_{-1}^{1} L_{m_\rho}(f_{m_\rho})^2\, dx \right]^{\frac{1}{2}} - 16, \end{aligned}$$

and by $(42\beta)$

$$I_{m_\rho} > 4^\rho - 16. \qquad \rho = 1, 2, 3, \cdots.$$

Hence Theorem III is established.

In conclusion, we take the opportunity of expressing our deep gratitude to Professor Fejér for his valuable help.

*Note added November 27, 1936.* The problem, whether the Lagrange-parabolas of a continuous function taken on a given B matrix can converge to any other value than the function itself is, as mentioned, undecided. Recently I. Marczinkievicz proved for the fundamental points given by the roots of

$$U_n(x) = 0 \left( U_n(\cos\theta) = \frac{\sin(n+1)\,\theta}{\sin\theta} \right)$$

that if the parabolas are convergent they always converge to the function itself. If the fundamental points are the roots of $T_n(x)$, we proved the same for $x \neq \frac{p}{q}\pi$, $(p, q) = 1$, $p \equiv q \equiv 1 \pmod 2$. On the other hand P. Erdös succeeded in showing that there exists a continuous function such that its Lagrange-parabolas taken upon this matrix converge to $+\infty$ at $x = \frac{\pi}{3}$.

BUDAPEST, HUNGARY.