ON PRIMITIVE ABUNDANT NUMBERS

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Let m be an integer and denote by $S(m)$ the sum of its divisors. Let

$$\sigma(m) = \frac{S(m)}{m}.$$

The number m is called a primitive abundant number (say p.a.n.) if

$$\sigma(m) \geqslant 2,$$

but, for $d \mid m$, $$\sigma(d) < 2.$$

Primitive abundant numbers were first discussed by Dickson†.

In a previous paper‡, I proved that the sum of the reciprocals of the p.a.n. is convergent by showing that $N(n)$, the number of p.a.n. not greater than n, satisfies

$$N(n) = O\left(\frac{n}{\log^2 n}\right).$$

I now prove that

$$\frac{n}{e^{c_1(\log n \log\log n)^{\frac{1}{2}}}} < N(n) < \frac{n}{e^{c_2(\log n \log\log n)^{\frac{1}{2}}}},$$

where the c's throughout denote constants.

First let us consider the upper bound of $N(n)$.

It is clear that, if a and b are different squarefree integers, then $\sigma(a) \neq \sigma(b)$; for, after performing the possible reductions, we obtain two irreducible fractions with different denominators.

We denote by the "squarefree" part of n the product of its prime factors which occur in n to the first power; *e.g.* the squarefree part of $2^3.3^2.5.7$ is 5.7.

We denote by the quadratic part of n the product of the prime factors whose exponents are greater than 1; *e.g.* the quadratic part of $2^3.3^2.5.7$ is $2^3.3^2$.

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† *American J. of Math.*, 35 (1913), 413–426.

‡ *Journal London Math. Soc.*, 9 (1934), 278–282.

For brevity, we write

$$x = (\log n \log\log n)^{\frac{1}{2}} \quad \text{and} \quad y = \left(\frac{\log n}{\log\log n}\right)^{\frac{1}{2}},$$

so that

$$xy = \log n \quad \text{and} \quad x/y = \log\log n.$$

We have to prove that

$$\frac{n}{e^{c_1 x}} < N(n) < \frac{n}{e^{c_2 x}}.$$

We shall show that it is sufficient to consider only the p.a.n. satisfying both the following conditions:

(1) if $m \leqslant n$, the quadratic part of m is less than $e^{\frac{1}{8}x}$,

(2) if $m \leqslant n$, the greatest prime factor of m is greater than e^{x}.

For we now prove that the number of integers less than or equal to n which do not satisfy these conditions is $O(n/e^{c_3 x})$.

It is evident that the quadratic part q of an integer is divisible by a square greater than or equal to $q^{\frac{2}{3}}$; for if

$$q = p_1^{2\alpha_1} p_2^{2\alpha_2} \ldots p_\mu^{2\alpha_\mu} r_1^{2\beta_1+1} \ldots r_\nu^{2\beta_\nu+1},$$

where the α's and β's are greater than or equal to 1, then q is divisible by the square

$$p_1^{2\alpha_1} p_2^{2\alpha_2} \ldots p_\mu^{2\alpha_\mu} r_1^{2\beta_1} \ldots r_\nu^{2\beta_\nu} \geqslant q^{\frac{2}{3}}.$$

Hence the integers not satisfying (1) are all divisible by a square greater than or equal to $e^{\frac{1}{12}x}$, and the number of such integers less than or equal to n is less than or equal to

$$\sum_{k^2 \geqslant e^{\frac{1}{12}x}} \frac{n}{k^2} \leqslant \frac{2n}{e^{\frac{1}{24}x}} = O\left(\frac{n}{e^{c_3 x}}\right),$$

where $c_3 < \frac{1}{24}$.

The integers not satisfying (2) may be divided into two classes.

In the first class are the integers for which the number of different prime factors is less than or equal to $\frac{1}{2}y$. We may suppose that their quadratic part is less than $e^{\frac{1}{8}x}$. Hence the number of such integers less than n is less than or equal to

$$e^{\frac{1}{2}(xy)+\frac{1}{8}x} = \sqrt{n}\, e^{\frac{1}{8}x} = O(n/e^{c_3 x}).$$

For the integers of the second class, the number of different prime factors is greater than $\frac{1}{2}y$. We denote the number of these integers by A and estimate its value as follows.

It is evident that every number containing at least k primes is divisible by an integer containing exactly k prime factors.

Let the integers less than or equal to n which contain exactly k prime factors be $a_1, a_2, \ldots, a_t$; then the number of integers containing at least k prime factors is less than or equal to

$$\frac{n}{a_1}+\frac{n}{a_2}+\ldots+\frac{n}{a_t}.$$

But
$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_t} \leqslant \frac{1}{k!}\Bigg(\sum_{\substack{p,\,d\\ p^d\leqslant n}} \frac{1}{p^d}\Bigg)^k.$$

Now
$$\sum_{\substack{p,\,d\\ p^d\leqslant n}} \frac{1}{p^d} \leqslant 2\log\log n.$$

Hence
$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_t} \leqslant \frac{2^k(\log\log n)^k}{k!},$$

and so the number of integers up to n containing at least k prime factors does not exceed

$$\frac{n\ 2^k(\log\log n)^k}{k!}.$$

Here $k=[\frac{1}{2}y]$, the square bracket denoting as usual the integral part, and, since

$$k!>\left(\frac{k}{e}\right)^k>\left(\frac{k}{3}\right)^k,$$

$$A<\frac{n\,2^{\frac{1}{2}y}(\log\log n)^{\frac{1}{2}y}\,3^{\frac{1}{2}y}}{(\frac{1}{2}y)^{\frac{1}{2}y}}<\frac{n\,e^{3y}\,e^{\frac{1}{2}y(\log\log\log n)}}{e^{\frac{1}{2}y\log\frac{1}{2}y}}.$$

Now
$$y\log y=\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{2}}\left(\frac{\log\log n-\log\log\log n}{2}\right)$$

$$>\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{2}}\frac{\log\log n}{4}=\tfrac{1}{4}x,$$

and so
$$A<\frac{n\,e^{3y+\frac{1}{2}y\log\log\log n}}{e^{\frac{1}{8}x}};$$

and, for sufficiently large n,

$$A<\frac{n}{e^{\frac{1}{16}x}}=O\left(\frac{n}{e^{c_3 x}}\right).$$

We shall now prove the following propositions concerning p.a.n. satisfying (1) and (2).

(a) The squarefree part of all such p.a.n. has a divisor between $\frac{1}{2}e^{\frac{1}{4}x}$ and $\frac{1}{2}e^{\frac{1}{2}x}$.

(b) If m is a p.a.n. satisfying (1) and (2), then

$$2 \leqslant \sigma(m) < 2 + \frac{2}{e^{x}}.$$

The proof of (a) requires the following lemma:

A p.a.n. satisfying (1) *and* (2) *has a divisor between* $\frac{1}{2}e^{\frac{1}{4}x}$ *and* $\frac{1}{2}e^{\frac{1}{2}x}$.

Let m be a p.a.n. satisfying the conditions (1) and (2) and let

$$m = uv,$$

where u contains only prime factors less than $\frac{1}{2}e^{\frac{1}{4}x}$, and v contains only prime factors greater than $\frac{1}{2}e^{\frac{1}{2}x}$.

If one of the prime factors of m lies between $\frac{1}{2}e^{\frac{1}{4}x}$ and $\frac{1}{2}e^{\frac{1}{2}x}$, it occurs only to the first power in consequence of (1), and so proposition (a) is evident.

Now $$u > \tfrac{1}{2}e^{\frac{1}{4}x},$$

for, if not, since m is a p.a.n.,

$$\sigma(u) < 2.$$

Then, since the difference between $\sigma(u)$ and 2 is at least $1/u$, we have

$$\sigma(u) \leqslant 2 - \frac{2}{e^{\frac{1}{4}x}}.$$

Further, $$\sigma(v) = \prod_{p \mid v} \left(1 + \frac{1}{p}\right),$$

since the prime factors of v occur only to the first power in consequence of (1).

Now every number less than n has at most $\log n$ prime factors, since, for large n,

$$([\log n])! > \left(\frac{[\log n]}{e}\right)^{[\log n]} > n.$$

Hence $$\sigma(v) < \left(1 + \frac{2}{e^{\frac{1}{2}x}}\right)^{\log n},$$

and so $$\sigma(v) < 1 + \frac{4 \log n}{e^{\frac{1}{2}x}}.$$

Consequently,

$$\sigma(m) = \sigma(u)\,\sigma(v)$$

$$< \left(2 - \frac{2}{e^{\frac{1}{4}x}}\right)\left(1 + \frac{4\log n}{e^{\frac{1}{2}x}}\right) < 2$$

for sufficiently large n; and this contradicts the hypothesis that m is abundant. Hence $u > \frac{1}{2}e^{\frac{1}{4}x}$.

Let $$u = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r},$$

where, from (1), $p_i^{a_i} < \frac{1}{2}e^{\frac{1}{4}x}$, and consider the numbers

$$p_1^{a_1},\ p_1^{a_1} p_2^{a_2},\ p_1^{a_1} p_2^{a_2} p_3^{a_3},\ \dots,\ p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}.$$

Evidently there is a λ such that

$$p_1^{a_1} p_2^{a_2} \dots p_\lambda^{a_\lambda} < \tfrac{1}{2}e^{\frac{1}{4}x} \leqslant p_1^{a_1} p_2^{a_2} \dots p_\lambda^{a_\lambda} p_{\lambda+1}^{a_{\lambda+1}}.$$

Since $$p_{\lambda+1}^{a_{\lambda+1}} < \tfrac{1}{2}e^{\frac{1}{4}x},$$

it follows that $$p_1^{a_1} p_2^{a_2} \dots p_{\lambda+1}^{a_{\lambda+1}} < \tfrac{1}{2}e^{\frac{1}{2}x},$$

i.e. the p.a.n. has a divisor in the desired interval.

Now, by (1), the quadratic part of the p.a.n. is less than $e^{\frac{1}{8}x}$ and so the squarefree part of it has a divisor between $\frac{1}{2}e^{\frac{1}{8}x}$ and $\frac{1}{2}e^{\frac{1}{2}x}$.

Proposition (*b*). Let p be the greatest prime factor of the p.a.n. m. From (1) and (2), we see that $p^2 \nmid m$, and so

$$\sigma(m) = \sigma\left(\frac{m}{p}\right)\left(1 + \frac{1}{p}\right).$$

Since m is a p.a.n.,

$$\sigma\left(\frac{m}{p}\right) < 2,$$

and so $$\sigma(m) < 2 + \frac{2}{p}.$$

Using (2), we have

$$2 \leqslant \sigma(m) < 2 + \frac{2}{e^x},$$

which proves (*b*).

We have now to prove that the number of p.a.n. satisfying conditions (1) and (2) is also

$$O\left(\frac{n}{e^{c_3 x}}\right).$$

We have seen that the p.a.n. satisfying the conditions (1) and (2) also satisfy the two propositions (*a*) and (*b*).

We now prove that the number of integers satisfying (1) and (2) and also (*a*) and (*b*) is

$$O\left(\frac{n}{e^{c_3 x}}\right).$$

This set of integers need not contain only p.a.n. We denote its elements by

$$C_1, C_2, \ldots, C_\tau.$$

We assert that $$\tau = O\left(\frac{n}{e^{c_3 x}}\right).$$

From (*a*), the squarefree part of each C_ω has a divisor D_ω lying between $\tfrac{1}{2}e^{\frac{1}{8}x}$ and $\tfrac{1}{2}e^{\frac{1}{4}x}$. Therefore

$$\frac{C_\omega}{D_\omega} < \frac{2n}{e^{\frac{1}{8}x}}.$$

We now show that $$\frac{C_{\omega_1}}{D_{\omega_1}} \neq \frac{C_{\omega_2}}{D_{\omega_2}},$$

i.e. that the number of integers C_ω/D_ω is less than $2n/e^{\frac{1}{8}x} = O(n/e^{c_3 x})$, and so the number of integers C_ω is $O(n/e^{c_3 x})$. Suppose that

$$\frac{C_{\omega_1}}{D_{\omega_1}} = \frac{C_{\omega_2}}{D_{\omega_2}};$$

then, evidently, $D_{\omega_1} \neq D_{\omega_2}$.

We prove the impossibility of the last equation by proving that

$$\sigma\left(\frac{C_{\omega_1}}{D_{\omega_1}}\right) \neq \sigma\left(\frac{C_{\omega_2}}{D_{\omega_2}}\right).$$

Now $$\sigma(C_{\omega_1}) = \sigma\left(\frac{C_{\omega_1}}{D_{\omega_1}}\right)\sigma(D_{\omega_1}),$$

and $$\sigma(C_{\omega_2}) = \sigma\left(\frac{C_{\omega_2}}{D_{\omega_2}}\right)\sigma(D_{\omega_2}).$$

Thus, if
$$\sigma\left(\frac{C_{\omega_1}}{D_{\omega_1}}\right)=\sigma\left(\frac{C_{\omega_2}}{D_{\omega_2}}\right),$$

then
$$\frac{\sigma(C_{\omega_1})}{\sigma(C_{\omega_2})}=\frac{\sigma(D_{\omega_1})}{\sigma(D_{\omega_2})}.$$

Since D_{ω_1} and D_{ω_2} are squarefree,
$$\sigma(D_{\omega_1})\neq\sigma(D_{\omega_2}),$$

and we may therefore suppose that
$$\frac{\sigma(D_{\omega_1})}{\sigma(D_{\omega_2})}>1,$$

for, if not, we consider its reciprocal $\dfrac{\sigma(D_{\omega_2})}{\sigma(D_{\omega_1})}$.

Now
$$\frac{\sigma(D_{\omega_1})}{\sigma(D_{\omega_2})}=\frac{S(D_{\omega_1})\,D_{\omega_2}}{S(D_{\omega_2})\,D_{\omega_1}},$$

and D_{ω_2} being a divisor of a p.a.n. is deficient. Hence
$$S(D_{\omega_2})<2D_{\omega_2},$$

and so the denominator of $\dfrac{\sigma(D_{\omega_1})}{\sigma(D_{\omega_2})}$ is less than $2D_{\omega_1}D_{\omega_2}$.

Hence
$$\frac{\sigma(D_{\omega_1})}{\sigma(D_{\omega_2})}\geqslant 1+\frac{1}{2\,D_{\omega_1}D_{\omega_2}}$$
$$>1+\frac{2}{e^x}>1+\frac{1}{e^x}.$$

But, from proposition (b), we see that
$$\frac{\sigma(C_{\omega_1})}{\sigma(C_{\omega_2})}<\frac{2+(2/e^x)}{2}=1+\frac{1}{e^x},$$

an evident contradiction.

Thus we have proved that
$$N(n)=O\left(\frac{n}{e^{c_3x}}\right),$$

and in fact
$$N(n)<\frac{n}{e^{\frac{1}{25}x}}.$$

Consider now the lower bound of $N(n)$.

The numbers $2^l p_1 p_2 \dots p_k$, $k \geqslant 2$, where $p_1 < p_2 < \dots < p_k$ are any k primes between $(k-1)\,2^{l+1}$ and $k 2^{l+1}$, are all p.a.n. (It will be shown by proper choice of k and l that the primes p actually exist.)

First we prove that they are abundant numbers. Now

$$\sigma(2^l p_1 p_2 \dots p_k) = \left(\frac{2^{l+1}-1}{2^l}\right)\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right) \dots \left(1+\frac{1}{p_k}\right)$$

$$\geqslant \left(\frac{2^{l+1}-1}{2^l}\right)\left(1+\frac{1}{k2^{l+1}-1}\right)\left(1+\frac{1}{k2^{l+1}-3}\right) \dots \left(1+\frac{1}{k2^{l+1}-2k+1}\right)$$

$$\geqslant \left(\frac{2^{l+1}-1}{2^l}\right)\left(1+\frac{1}{k2^{l+1}-1}+\frac{1}{k2^{l+1}-3}+\dots+\frac{1}{k2^{l+1}-2k+1}\right)$$

$$\geqslant \frac{2^{l+1}-1}{2^l}\left(1+\frac{k^2}{k^2\,2^{l+1}-k^2}\right),$$

since
$$\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_k} \geqslant \frac{k^2}{x_1+x_2+\dots+x_k},$$

if the x's are positive; and so

$$\sigma(2^l p_1 p_2 \dots p_k) \geqslant \left(\frac{2^{l+1}-1}{2^l}\right)\left(1+\frac{1}{2^{l+1}-1}\right)$$

$$= \frac{2^{l+1}-1}{2^l}\,\frac{2^{l+1}}{2^{l+1}-1} = 2.$$

We now prove that they are p.a.n. For this we must prove that

$$\sigma(2^{l-1} p_1 p_2 \dots p_k) < 2,$$

and
$$\sigma(2^l p_1 p_2 \dots p_{k-1}) < 2.$$

Now
$$\sigma(2^{l-1} p_1 p_2 \dots p_k) < \sigma(2^l p_1 p_2 \dots p_{k-1}),$$

i.e.
$$\frac{2^l-1}{2^{l-1}}\left(1+\frac{1}{p_k}\right) < \frac{2^{l+1}-1}{2^l},$$

if
$$1+\frac{1}{p_k} < \frac{2^{l+1}-1}{2^{l+1}-2} = 1+\frac{1}{2^{l+1}-2},$$

i.e., if
$$p_k > 2^{l+1}-2.$$

This is true since
$$p_k > (k-1)\,2^{l+1}.$$

Thus we need only prove that

$$\sigma(2^l p_1 p_2 \dots p_{k-1}) < 2,$$

since the omission of any of the other factors $p_1, p_2, \dots, p_k$ gives a smaller σ.

Now

$$\sigma(2^l p_1 p_2 \dots p_{k-1})$$

$$\leqslant \left(\frac{2^{l+1}-1}{2^l}\right)\left(1+\frac{1}{(k-1)2^{l+1}+1}\right)\left(1+\frac{1}{(k-1)2^{l+1}+3}\right)\dots\left(1+\frac{1}{(k+1)2^{l+1}+2k-3}\right)$$

$$< \left(\frac{2^{l+1}-1}{2^l}\right)\left(1+\frac{1}{(k-1)\,2^{l+1}}\right)^{k-1}$$

$$< \left(\frac{2^{l+1}-1}{2^l}\right)\left\{1+\frac{1}{2^{l+1}}+\frac{1}{2!}\left(\frac{1}{2^{l+1}}\right)^2+\frac{1}{3!}\left(\frac{1}{2^{l+1}}\right)^3+\dots\right\}$$

$$< \left(\frac{2^{l+1}-1}{2^l}\right)\left(1+\frac{1}{2^{l+1}}+\frac{1}{2^{2l+2}}\right)$$

$$= \left(\frac{2^{l+1}-1}{2^l}\right)\left(\frac{2^{2l+2}+2^{l+1}+1}{2^{2l+2}}\right) = \frac{2^{3l+3}-1}{2^{3l+2}} \leqslant 2.$$

Now choose l and k so that

$$e^{x-4} < 2^l < e^{x-3},$$

$$k = [y-2].$$

Clearly $l > k$, since $x > y$.

In this case, the p.a.n. of the form $2^l p_1 p_2 \dots p_k$ are all less than

$$e^{x-3}(y-2)^{y-2}\, e^{(x-2)(y-2)}.$$

Now $$(y-2)^{y-2} < y^y = e^{y \log y} < e^x.$$

Hence $$e^{x-3}(y-2)^{y-2}\, e^{(x-2)(y-2)} < e^{2x-3+xy-2x-2y+4}$$

$$= e^{xy-2y+1} < e^{xy} = n.$$

Consequently the p.a.n. in question are all less than n. We now estimate their number.

By the more exact form of the prime number theorem we see that the number of primes between $(k-1)\,2^{l+1}$ and $k2^{l+1}$ is greater than

$$\frac{e^{x-4}}{2x} > (1\,.\,1)\,\frac{e^{x-5}}{x}.$$

For $$\pi(u) = \int_2^u \frac{du}{\log u} + O\left(\frac{u}{(\log u)^h}\right)$$

$$= \frac{1}{\log 2} + \frac{1}{\log 3} + \dots + \frac{1}{\log u} + O\left(\frac{u}{(\log u)^h}\right)$$

for every h.

Take $h = 4$, then

$$\pi\{(k-1)\,2^{l+1}\} = \frac{1}{\log 2} + \frac{1}{\log 3} + \dots + \frac{1}{\log (k-1)\,2^{l+1}} + O\left(\frac{2^l}{l^2}\right)$$

and

$$\pi(k2^{l+1}) = \frac{1}{\log 2} + \frac{1}{\log 3} + \dots + \frac{1}{\log (k-1)\,2^{l+1}} + \dots + \frac{1}{\log k2^{l+1}} + O\left(\frac{2^l}{l^2}\right).$$

Hence

$$\pi(k2^{l+1}) - \pi\{(k-1)2^{l+1}\} = \frac{1}{\log\{(k-1)2^{l+1}+1\}} + \dots + \frac{1}{\log k2^{l+1}} + O\left(\frac{2^l}{l^2}\right),$$

i.e.

$$\pi(k2^{l+1}) - \pi\{(k-1)2^{l+1}\} > \frac{2^{l+1}}{\log (k2^{l+1})} + O\left(\frac{2^l}{l^2}\right)$$

$$= \frac{2^{l+1}}{\log k + (l+1)\log 2} + O\left(\frac{2^l}{l^2}\right)$$

$$> \frac{2^{l+1}}{l+1} + O\left(\frac{2^l}{l^2}\right) > \frac{2^l}{l+1}$$

$$> \frac{e^{x-4}}{2x} > \frac{1.1\,e^{x-5}}{x}.$$

Hence

$$N(n) > \binom{[1.1\,e^{x-5}/x]}{[y-2]} > \frac{e^{(x-5)(y-2)}}{x^y (y-2)^{y-2}},$$

since

$$\binom{n}{k} = \frac{n}{k}\,\frac{(n-1)}{k-1} \dots \frac{n-k+1}{1} > \left(\frac{n}{k}\right)^k,$$

i.e.

$$N(n) > \frac{e^{xy}}{x^y y^y e^{2x+5y}} = \frac{n}{(\log n)^y e^{2x+5y}}$$

$$> \frac{n}{e^{y \log\log n + 7x}} = \frac{n}{e^{8x}}.$$

Thus we have proved that, for sufficiently large n,

$$\frac{n}{e^{8(\log n \log\log n)^{\frac{1}{2}}}} < N(n) < \frac{n}{e^{\frac{1}{25}(\log n \log\log n)^{\frac{1}{2}}}}.$$

The difference between the constants may be reduced by longer calculations.

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