Polychromatic colorings of arbitrary rectangular partitions

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Abstract

A general (rectangular) partition is a partition of a rectangle into an arbitrary number of non-overlapping subrectangles. This paper examines vertex colorings using four colors of general partitions where every subrectangle is required to have all four colors appear on its boundary. It is shown that there exist general partitions that do not admit such a coloring. This answers a question of Dimitrov et al. [3]. It is also shown that the problem to determine if a given general partition has such a 4-coloring is **NP**-Complete. Some generalizations and related questions are also treated.

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1 Introduction

A k-coloring of the vertices of a plane graph (a graph drawn in the plane with no crossing edges) is *polychromatic* if on every face all k colors appear at least once (with the possible exception of the outer face). The *polychromatic number* of a plane graph G is the *maximum* number k such that G admits a polychromatic k-coloring; we denote this number by $\chi_f(G)$. For an introduction to polychromatic colorings see, for example, the introduction of [1] or [3]. We restrict ourselves to a brief general discussion of this topic and mention some results. Alon et al. [1] showed that if g is the length of a shortest face of a plane graph G, then $\chi_f(G) \geq \lfloor (3g-5)/4 \rfloor$ (clearly $\chi_f(G) \leq g$) and showed that this bound is sufficiently tight. Mohar and Škrekovski [11], using the four-color theorem, proved that every simple plane graph admits a polychromatic 2-coloring. Later Bose et al. [2] proved this result without the use of the four-color theorem. Horev and Krakovski [8] proved that every plane graph of degree at most 3 other than K_4 admits a polychromatic 3-coloring. Horev et al. [6] proved that every 2-connected cubic bipartite plane graph admits a polychromatic 4-coloring. This result is tight, since any such graph must contain a face of size four.

In a series of papers the following special case was considered. A *rectangular partition* is a partition of an axis-parallel rectangle into an arbitrary number of non-overlapping axis-parallel rectangles such that no four rectangles meet at a common point. One may view a rectangular partition as a plane graph whose vertices are the corners of the rectangles and edges are the line segments connecting these corners. Such a graph always contains a cycle of length 4, thus $\chi_f(G) \leq 4$. Dinitz et al. [4] proved that every rectangular partition admits a polychromatic 3-coloring.

Before continuing, let us introduce two related coloring notions. We define a *weak rectangle-respecting k-coloring* of a rectangular partition as a k-coloring of the vertices as follows: if $k \leq 4$, then for each rectangle S exactly k colors must appear on the vertices of the *boundary* of S and if $k \geq 4$, then for each rectangle S at least 4 colors must appear on the vertices of the *boundary* of S. Note that in the first case this definition corresponds exactly with the notion of a polychromatic k-coloring as every rectangular partition contains a rectangle (i.e. a cycle) with exactly four vertices on its boundary. Furthermore, we define a *strong rectangle-respecting k-coloring* of a rectangular partition as a k-coloring of the vertices as follows: if $k \leq 4$, then for each rectangle S exactly k colors must appear on the *corners* of S and if $k \geq 4$, then for each rectangle S at least 4 colors must appear on the *corners* of S. We remark that the case $k = 4$ is consistent with both of the

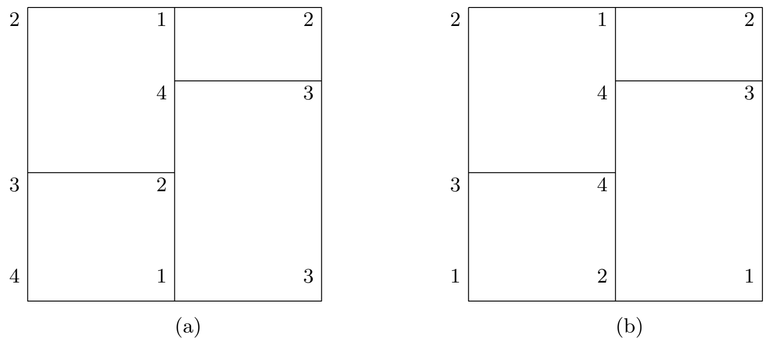

Figure 1: (a) weak rectangle-respecting 4-coloring and (b) strong rectangle-respecting 4-coloring of a guillotine-partition

respective cases in both definitions.

Note that a strong rectangle-respecting coloring is necessarily stronger than a weak rectangle-respecting coloring as the boundary of a rectangle includes its four corners. Thus, in the case $k \leq 4$, a strong rectangle-respecting k-coloring gives a polychromatic k-coloring. See Figure 1 for examples.

For $k \geq 4$, it is clear that the existence of a weak (resp. strong) rectangle-respecting k-coloring implies the existence of a weak (resp. strong) rectangle-respecting $(k+1)$-coloring (just ignore additional colors). Furthermore, for $k \leq 4$ the existence of a weak (resp. strong) rectangle-respecting k-coloring implies the existence of a weak (resp. strong) rectangle-respecting $(k-1)$-coloring. Thus, we should focus our attention on finding weak and strong rectangle-respecting k-colorings for k as close to 4 as possible.

A rectangular partition obtained by recursively cutting a rectangle into two subrectangles by either a vertical or a horizontal line is called a *guillotine-partition*. Horev et al. [7] proved that every element of this subclass of rectangular partitions admit a strong rectangle-respecting 4-coloring (and hence a polychromatic 4-coloring). Recently, Dimitrov et al. [3] proved that any rectangular partition admits a strong rectangle-respecting 4-coloring, using a theorem about plane graphs. Moreover, the second author of the present paper proved a natural generalization of this result for n dimensional guillotine-partitions [9].

In the above papers the results are restricted to partitions where no four rectangles are allowed to meet at a common corner. In this case the strongest possible statement holds and is proved, namely the existence of a strong rectangle-respecting 4-coloring (i.e. a polychromatic 4-coloring). If we allow four rectangles to meet at a common corner in the partition described above, then we have a *general partition*. The notions of weak and strong rectangle-respecting k-colorings are naturally extendable to this more general situation. In [3] the authors construct a rectangular partition that has no strong rectangle-respecting 4-coloring. The authors continue by asking if every general partition has a weak rectangle-respecting 4-coloring. We answer this question in the negative by constructing a general partition that has no weak rectangle-respecting 4-coloring. In fact, the constructed general partition is also a guillotine-partition and therefore answers the question negatively even in this special case.

Theorem 1. *There exists a general partition with no weak rectangle-respecting* 4*-coloring (i.e. no polychromatic* 4*-coloring).*

Furthermore, a simple characterization of weak rectangle-respecting 4-coloring is unlikely.

Theorem 2. *Deciding whether a general partition admits a weak rectangle-respecting* 4*-coloring is* **NP***-complete.*

The negative answer from Theorem 1 gives rise to the problem of finding the largest k for which such coloring always exists. We show that such a coloring with 3 colors always exists. Note that the result of Dinitz et al. [4] follows.

Proposition 3. *Every general partition admits a weak rectangle-respecting* 3*-coloring (i.e. a polychromatic* 3*-coloring).*

Thus the following proposition answers the remaining cases.

Proposition 4. *Every general partition admits a weak rectangle-respecting* 5*-coloring.*

Now let us turn our attention to the remaining questions for strong rectangle-respecting k-colorings. We show the existence of strong rectangle-respecting 2- and 6-colorings.

Proposition 5. *Every general partition admits a strong rectangle-respecting* 2*-coloring.*

Proposition 6. *Every general partition admits a strong rectangle-respecting* 6*-coloring.*

Note that simple coloring algorithms will follow from the proofs of the above propositions. The existence of strong rectangle-respecting 3- and 5-colorings remains unknown.

Problem 7. *Does every general partition admit a strong rectangle-respecting* 3*-coloring?*

Problem 8. *Does every general partition admit a strong rectangle-respecting* 5*-coloring?*

For simplicity, in the remainder of the paper we will use the terms partition and general partition interchangeably.

2 Weak rectangle-respecting 4-colorings

In this section we consider partitions where four rectangles are allowed to meet in a common point (i.e. a general partition). We want to find a 4-coloring of a given partition such that for any face r all four colors appear on the vertices on the boundary of r. Denote by $\mathcal{G}$ the 3×3 grid (i.e. four squares) and by T the partition obtained from a 3×3 grid by merging the upper two squares (see Figure 2). When referring to a *side* of the partition $\mathcal{G}$ or T we mean the set of vertices on the corresponding vertical or horizontal boundary of the partition (e.g. left, right, upper, lower). (Here and throughout the paper when we refer to something being colored with k colors, we mean *exactly* k distinct colors.) We begin with some simple observations.

Observation 9. *If a weak rectangle-respecting* 4*-coloring of the* 3×3 *grid* $\mathcal{G}$ *assigns three colors to the left (resp. upper) side of* $\mathcal{G}$*, then the same three colors appear on the right (resp. lower) side of* $\mathcal{G}$*.*

Observation 10. *A weak rectangle-respecting* 4*-coloring of a* 3×3 *grid* $\mathcal{G}$ *cannot simultaneously assign three colors to the left (or right) and lower (or upper) sides.*

Observation 11. *A weak rectangle-respecting* 4*-coloring of* T *assigns three colors to the left side or right side of* T*.* (See Figure 2.)

Let us define a new partition $\mathcal{Q}$ as follows: start with a 7×7 grid, first merge the four central squares, then for each side of this new center square merge the two smaller squares adjacent to that side. In this way we obtain a partition that contains four copies of $\mathcal{G}$ and four rotations of T. See Figure 3 for an illustration of $\mathcal{Q}$.

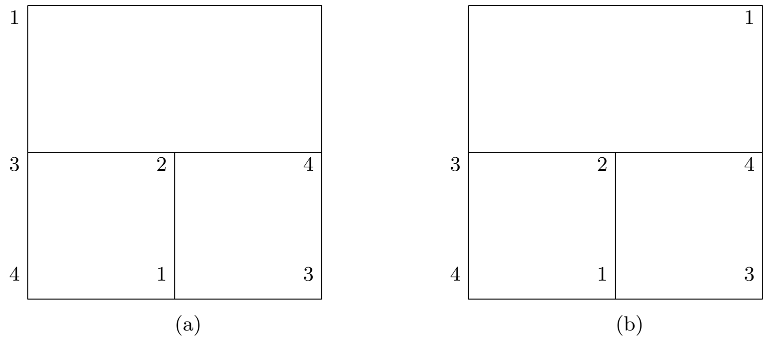

Figure 2: The partition T and two different colorings

Claim 12. *Let G_1 (resp. G_2) be the 3×3 grid in the upper-left (resp. upper-right) corner of $\mathcal{Q}$. A weak rectangle-respecting 4-coloring of the partition $\mathcal{Q}$ must assign three colors to either the upper side of G_1 or the upper side of G_2.*

Proof. Let G_3 (resp. G_4) be the 3×3 grid in the lower-left (resp. lower-right) corner of $\mathcal{Q}$ (see Figure 3). Assume that neither the upper side of G_1 nor the upper side of G_2 are assigned three colors. By Observation 9 neither the lower side of G_1 nor the lower side of G_2 are assigned three colors. By Observation 11 the upper sides of G_3 and G_4 are each assigned three colors. Finally, by Observation 10 the right side of G_3 has two colors and the left side of G_4 has two color. However, now we have colored the right and left sides of the partition T on the bottom of $\mathcal{Q}$ in a way that contradicts Observation 11. □

Note that a similar claim holds for each side of $\mathcal{Q}$ as we can simply rotate $\mathcal{Q}$ and follow the proof of Claim 12. Let us define a new partition $\mathcal{C}$ as follows: start with a 3×3 grid and embed a 7×7 grid in the upper-right square, a copy of $\mathcal{Q}$ in the upper-left square and a copy of $\mathcal{Q}$ in the lower-right square. See Figure 4 for an illustration. We will show that partition $\mathcal{C}$ has no weak rectangle-respecting 4-coloring thus proving Theorem 1.

Proof of Theorem 1. Let Q_1 be the upper-left copy of $\mathcal{Q}$ and let Q_2 be the lower-right copy of $\mathcal{Q}$. By applying Claim 12 to Q_2 we find 3 consecutive vertices on the lower side of the 7×7 grid with three colors. By applying Claim 12 to a -90^o rotation of Q_1 we find 3 consecutive vertices on the right side of the 7×7 grid with three colors. By application of Observation 9 it is

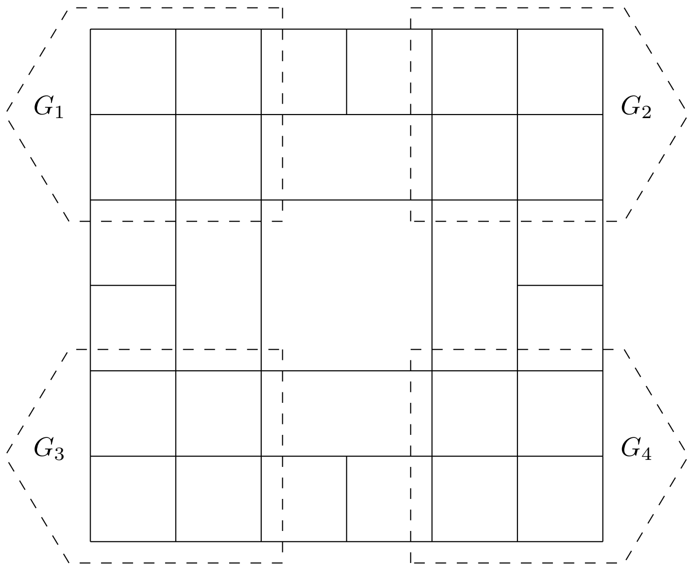

Figure 3: The partition $\mathcal{Q}$ and subpartitions G_1, G_2, G_3, G_4

easy to see that we can find a 3×3 grid in the 7×7 grid that has three colors on both the lower side and right side. This contradicts Observation 10. □

The authors have constructed several other partitions that have no weak rectangle-respecting 4-coloring. The smallest known construction is made up of 46 rectangles and has 65 vertices. It would be interesting to find smaller examples.

We now turn to the proof of Theorem 2. We omit many small details of the proof as a rigorous proof would be rather lengthy. Our focus will be to present the main steps from which the interested reader should easily be able to reconstruct the complete proof.

Proof of Theorem 2. The proof is by reduction from PLANAR–3SAT which was proved to be NP-complete by Lichtenstein [10]. We say that a conjunctive normal form (CNF) is planar if there is a planar bipartite graph with vertex set X, Y such that X is the set of variables and Y is the set of clauses and there is an edge between a variable x and a clause y if and only if x is contained in y in the CNF. [1] PLANAR–3SAT is the decision problem

[1] In fact Lichtenstein proved a somewhat stronger statement and sometimes this weaker version is denoted by PLANAR*–3SAT.

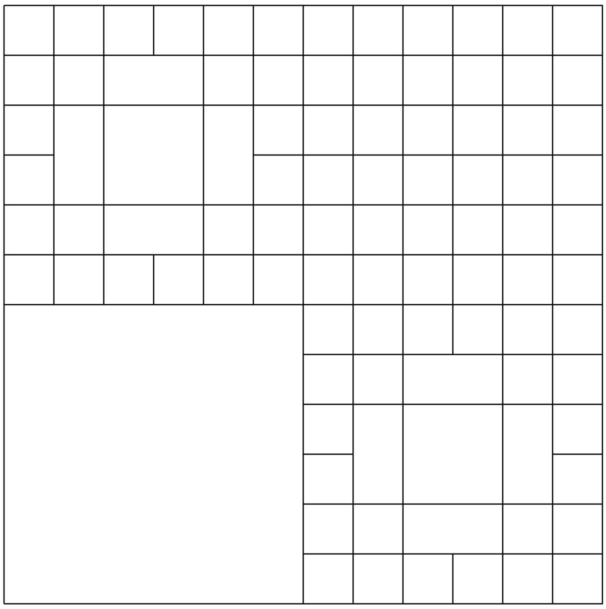

Figure 4: $\mathcal{C}$, the counterexample proof of Theorem 1

of whether a given planar conjunctive normal form in which every clause consists of 3 literals is satisfiable or not.

To reduce from Planar–3SAT for a given planar CNF G we must construct a partition R_G that has a weak rectangle-respecting 4-coloring if and only if G is satisfiable. To do this we will assume we are given G in its planar graph form (G will simultaneously denote the CNF and its representation as a planar graph). Then we will build a partition R_G that resembles G in such a way that it will be easy to confirm that R_G has a weak rectangle-respecting 4-coloring if and only if G is satisfiable. The partition R_G will be made up of subpartitions corresponding to the variables and clauses of G. We will use the term *component* to refer to a subpartition of R_G that represents a variable or clause of G. The construction of R_G will be split into four parts: "edges", "clauses", "variables" and "putting it together".

Edges

Let e be an edge in the planar CNF G. In the partition R_G we will construct a subpartition representing e that connects the appropriate clause and variable component. Such a subpartition is called a *grid-edge*. The grid-edge

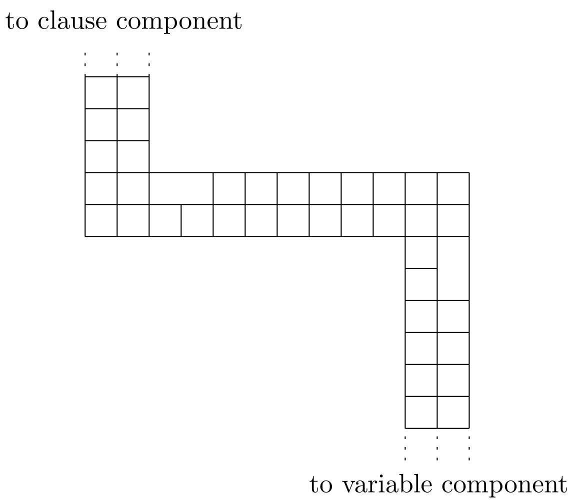

Figure 5: a grid-edge connecting a clause to a variable

representing e will be an alternating series of vertical and horizontal grids of width 3 and of appropriate length, one end of a grid-edge will be attached to a variable component (the *variable end*) and the other end attached to a clause component (the *clause end*). Vertical and horizontal subpartitions of a grid-edge are connected by a suitable rotation of the partition T as illustrated in Figure 5. Note that once a grid-edge is fixed between two subpartitions, we can add an arbitrary number of extra turns within the grid-edge without disrupting the start or end of the grid-edge. To give us sufficient freedom to choose the coloring of a grid-edge it will be necessary to assume that every grid-edge in R_G has at least six turns.

Grid-edges allow us to force certain arrangements of colors in two different parts of the partition. The following observations are easy consequences of the observations stated earlier in this section.

Observation 13. *Let E be a grid-edge with a weak rectangle-respecting 4-coloring. If the 3 vertices on the variable end of E are colored with two colors, then the 3 vertices on the clause end of E are also colored with two colors.*

Note that because grid-edges are not symmetric, Observation 13 is not true if we exchange the terms variable component and clause component. It will also be necessary to consider the case when the 3 vertices on the variable end of a grid-edge are colored with three colors.

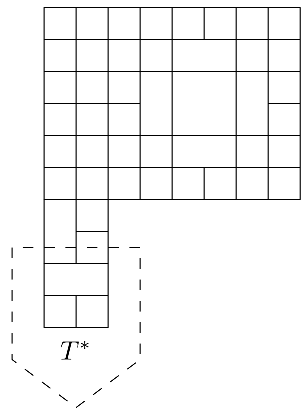

Figure 6: a clause component S_c and subpartition T^*

Observation 14. *If E is a grid-edge, then there exists a weak rectangle-respecting 4-coloring of E such that the 3 vertices on the variable end of E are colored with three colors and the 3 vertices on the clause end of E are also colored with three colors.*

Moreover, in both observations we can color a grid-edge in such a way that the clause end can get any (legal) assignment of colors. In particular, each turn on a grid-edge gives an additional freedom of choice of colors (due to the T partition). After six such turns we will have sufficient freedom to choose any assignment. (We omit the proof of this claim; although simple it involves a lengthy case analysis.) This detail will allow us to avoid potential conflicts when coloring the entire partition R_G.

Clauses

Let c be a clause of the planar CNF G. Every clause has degree 3 in G (i.e. every clause contains 3 literals). We construct a copy of the subpartition S_c in Figure 6 for each clause c. In particular, the subpartition S_c will be connected to each subpartition representing a variable v in c.

Let T^* be the subpartition of S_c (Figure 6) consisting of a copy of T with an extra vertex on the top edge. By examining Figure 6 and applying Claim 12 and the observations in Section 2 it can be seen that the 3 vertices on the top of T^* must get three colors in a weak rectangle-respecting 4-coloring of S_c. Furthermore, among the three remaining sides of T^* at least one side must have three colors among its vertices. In particular, any one side or any two sides or all three sides may have three colors among their

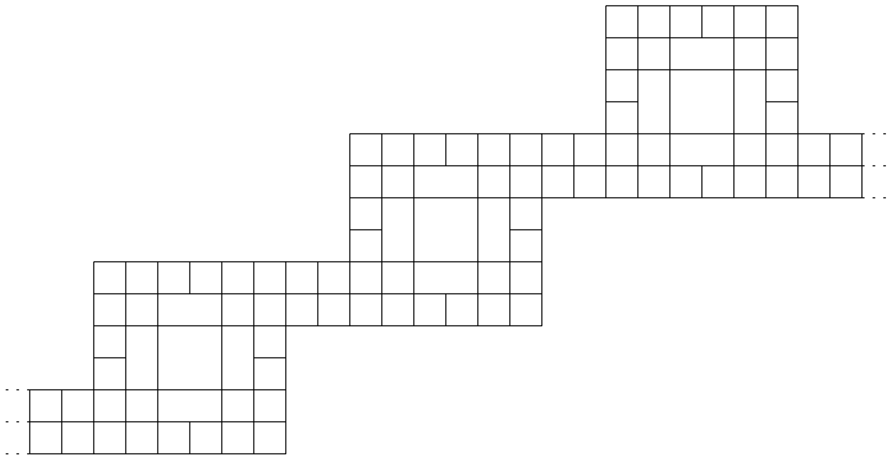

Figure 7: a part of a variable component

respective 3 vertices. Any of these cases will correspond to the clause being TRUE.

Ultimately, the left, right and bottom sides of T^* will be connected by grid-edges to variable components (see Figure 8).

Variables

Let v be a variable of the planar CNF G and let the degree (i.e. the number of clauses that contain v or the complement of v) be $d(v)$. We will construct a subpartition S_v of R_G corresponding to v. The subpartition consists of $d(v)$ copies of the partition $\mathcal{Q}$. (The subpartitions G_1, G_2, G_3, G_4 of $\mathcal{Q}$ are defined in Figure 3.) These copies of $\mathcal{Q}$ are arranged in a sequence as shown in Figure 7. This sequence forces that in a weak rectangle-respecting 4-coloring of R_G either:

Case 1 For every copy of $\mathcal{Q}$ in S_v the upper side G_1 has three colors and the upper side of G_2 has two colors.

Case 2 For every copy of $\mathcal{Q}$ in S_v the upper side of G_2 has three colors and the upper side of G_1 has two colors.

The first case will correspond to the variable v being TRUE and the second case will correspond to v being FALSE. Each copy of a $\mathcal{Q}$ in S_v will correspond to exactly one clause that contains v (or $\bar{v}$). If v is unnegated in a clause c then we will attach one end of a grid-edge to the upper side of

G_1 of the appropriate instance of $\mathcal{Q}$ and the other end will be attached to S_c. If v is negated in the clause c then we will attach one end of a grid-edge to the upper side of G_2 of the appropriate instance of $\mathcal{Q}$ and the other end will be attached to S_c.

Putting it together

First let us draw the planar graph G in such a way that all vertices lay on an $n^c \times n^c$ sized grid for $n = |V(G)|$ and a suitable constant c. Such a drawing follows easily from any grid representation theorem (see e.g. [5]). This drawing ensures that the size of the new graph is polynomial in n. Now replace every variable vertex v of G with S_v, replace every clause vertex c of G with S_c and replace each edge of G with a grid-edge connecting the appropriate parts of S_v and S_c. See Figure 8.

To complete the construction of the general partition R_G we need to add rectangles to embed what we have constructed so far in a bounding rectangle. Let us not go into the complete details, but this can be done easily. We must guarantee that these new rectangles can be colored appropriately regardless of how the grid-edges, clause components and variable components are colored. If we partition each new rectangle as in Figure 9 (or a rotation) we can avoid any potential conflicts. Thus we can ignore these new rectangles when discussing the colorablity of R_G.

Now let us show that the satisfiability of G is equivalent to the existence of a weak rectangle-respecting 4-coloring for R_G. First we assume that R_G has a weak rectangle-respecting 4-coloring and show that G has a satisfying assignment. Under such a coloring of R_G each variable component either satisfies Case 1 or Case 2 in the variables section above. If a variable component S_v satisfies Case 1 then fix v as TRUE in G. If S_v satisfies Case 2 then fix v as FALSE in G. Let us confirm that such an assignment of TRUE/FALSE values satisfies G. In every clause c at least one literal should be TRUE. Let us assume to the contrary that there is a clause c with all three literals FALSE. For the sake of simplicity, let us assume that all three literals in c are unnegated variables (it is easy to modify the argument if one or more of the literals are negated variables). Therefore, all three variable components connected by grid-edges to S_c are colored so that Case 2 is satisfied. This implies that for each grid-edge E attached to S_c, the three vertices on the variable end of E are colored with two colors. By Observation 13 the 3 vertices of the clause end of each grid-edge connected to S_c are each also colored with two colors. This contradicts the required coloring of the subpartition T^* described in the clauses section. Thus, every

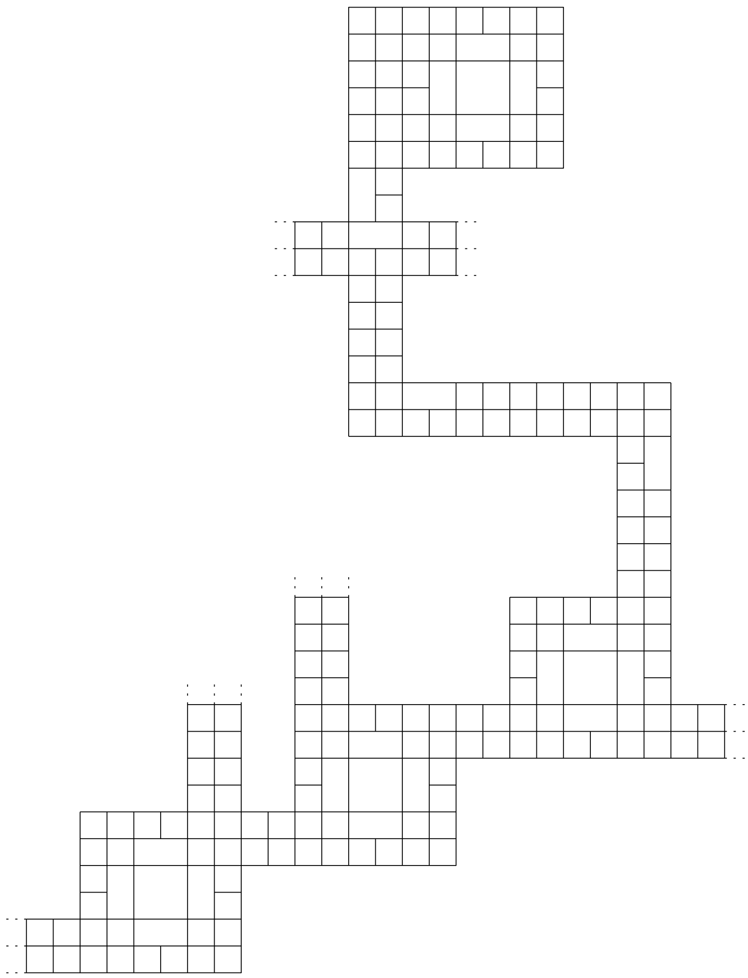

Figure 8: connecting a clause component to a variable component

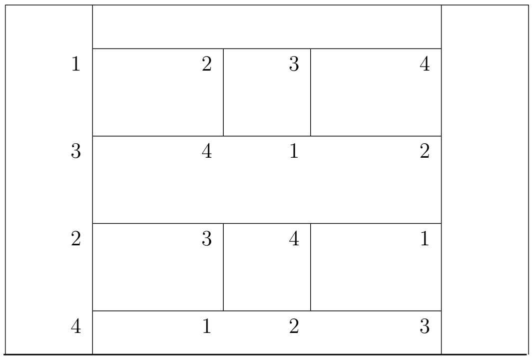

Figure 9: partition and coloring of a rectangle

clause has a true literal and G has a satisfying assignment.

Now we assume that G has a satisfying assignment and show that R_G has a weak rectangle-respecting 4-coloring. If v is assigned TRUE in G, then color S_v as described in Case 1 in the variables section above. If v is FALSE in G, then color S_v as described in Case 2. Each variable component can be colored independently in this way. Furthermore, the coloring of the variable components determines the arrangement of colors on the variable end of each grid-edge. Now let us confirm that each clause component can be colored. Each clause contains a TRUE literal, thus for each clause component there is a grid-edge connected to it that has three colors on the variable end. By Observation 14, there is a coloring of this grid-edge such that the clause end is also colored with three colors. To actually color the three incoming grid-edges to a clause we must be careful to avoid conflicting colors. This can be done because of the remark ending the edges section. This completes the sketch of the proof. □

3 Weak rectangle-respecting 3- and 5-colorings

This section is concerned with the proof of Proposition 3 and Proposition 4 i.e. any partition admits a weak rectangle-respecting 3- and 5-coloring.

Consider a partition in the coordinate plane. A partition consists of axis-parallel rectangles, so this gives a simple ordering of the vertices based on their two coordinates. In particular, let us arrange vertices from smallest to largest x coordinate then from largest to smallest y coordinate i.e. from left to right then top to bottom. We refer to this ordering as the *upper-left order* of the vertices.

Proposition 3. *Every general partition admits a weak rectangle-respecting* 3*-coloring (i.e. a polychromatic* 3*-coloring).*

Proof. Let R be a partition. We will greedily 3-color the vertices of R. Any vertex v of the partition can have at most one neighbor with smaller x-coordinate and at most one neighbor with larger y-coordinate. When coloring R, we will always maintain the property that v is colored with a color not used by either of these (at most) two neighbors.

Consider the vertices of R to be arranged in upper-left order (as described above). For each subrectangle of R we will ensure that it has 3 colors on its boundary after we color the vertex in its lower right corner.

Let v be the pending vertex to be colored. If v has no previously colored neighbors (i.e. it is the upper-left corner or R), then choose any color for v.

If v has exactly one previously-colored neighbor w then color v with a color different from w. If v has two previously-colored neighbors, say x and y, then v, x and y are on the boundary of a common rectangle. Denote this rectangle by r and note that v is the bottom-right corner of r.

If x and y are colored with different colors then color v with the color unused by x and y. Thus, the rectangle r will have 3 different colors on its boundary.

If y and x are colored with the same color then we will color v with a color different from that used on x and y. But x must have an already colored neighbor w on the boundary of rectangle r. We maintained the property that a vertex is colored with a different color from its previously colored neighbors, thus w and x have different colors. Color v with the color unused by x and w. Thus, the rectangle r will have 3 different colors on its boundary.

In this way every rectangle of R will include three different colors among the vertices on its boundary. □

Proposition 4. *Every general partition admits a weak rectangle-respecting* 5*-coloring.*

Proof. Let R be a partition. We will 5-color the vertices of R with the following algorithm. Consider the vertices of R in upper-left order. Let v be the pending vertex to be colored and (when they exist) let x be the neighbor to the left of v and y be the neighbor above v. Further, (when they exist) let A_v be the rectangle with v as its lower-right corner and let B_v be the rectangle with v as its lower-left corner and let w be the neighbor of y (other than v) in B_v (w may be above y or to the right of y). By the upper-left

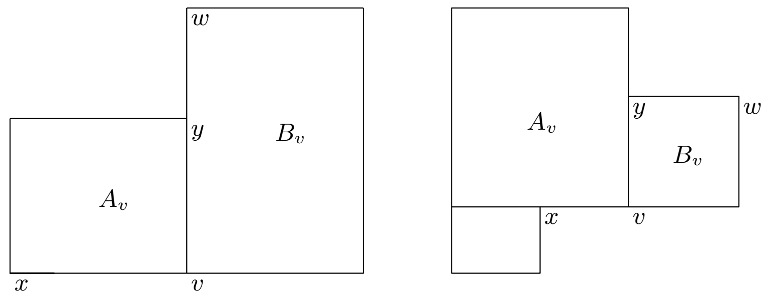

Figure 10: two examples of the pending vertex v

order, each vertex x, y and w is already colored (or does not exist). See Figure 10 for examples.

We will choose a color for v that is different from:

1. the color of x
2. the color of y
3. the color of w
4. three colors used on the vertices of A_v (including the one or two colors used on x and y)

Note that for condition 4 we must justify the existence of three colors on A_v before v is colored.

Observe that conditions 1 and 2 imply that the coloring of R (as a graph) is proper and thus each rectangle gets at least 2 colors. Then condition 3 implies that the rectangle B_v gets at least 3 colors, as v, y and w all have different colors. Furthermore, there exists a vertex v' (appearing before v in the upper-left ordering) such that $A_v = B_{v'}$ and thus before coloring v, A_v is already colored with at least 3 colors. This justifies the statement of condition 4. Thus, the four conditions imply that the vertices of A_v will be colored with at least 4 colors. For every rectangle C, there exists a vertex c such that $C = A_c$ and thus every C is colored with at least 4 colors.

For a vertex v, the four conditions forbid at most four colors for v and thus 5 colors are enough to complete the coloring of R. □

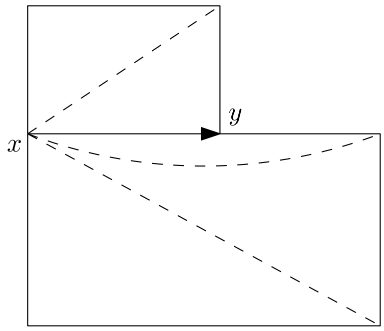

Figure 11: orientation of edge xy

4 Strong rectangle-respecting colorings

In this section we either want to find the minimal number $k \geq 4$ or maximal number $k < 4$ of colors such that every partition can be colored with k colors such that $\min\{k, 4\}$ colors appear on the 4 *corners* of every rectangle of the partition.

Proposition 5. *Every general partition admits a strong rectangle-respecting* 2*-coloring.*

Proof. For a given partition R let us color the vertices in upper-left order with two colors. Let v be the pending vertex to be colored. Only vertices above and to the left of v are already colored. Thus, only the rectangle that has v in its lower right corner may have three previously colored corners. If the three previously colored corners have the same color, then choose the other color for v. Otherwise choose any color for v. After coloring all vertices in this way clearly no rectangle will have a single color among its four corners. $\square$

A simple greedy algorithm similar to those used for weak rectangle-respecting 5-colorings shows that for a partition there is always a strong rectangle-respecting 7-coloring. However, with a little extra care, we can do better.

Proposition 6. *Every general partition admits a strong rectangle-respecting* 6*-coloring.*

Proof. For a given partition R, let G be the graph with the vertex set of R, where xy is an edge of G if and only if x and y are corners of the same rectangle in R. Clearly, the proposition is proved if we can find a proper 6-coloring of G. First we will color the vertices where four rectangles meet. We use a greedy algorithm with the vertices in upper-left order. Every vertex has at most four previously colored neighbors, hence six colors are (more than) enough to properly color such vertices.

Denote by W the set of all so-far uncolored vertices of R. A vertex in W is the corner of at most two rectangles, thus has degree at most six in the graph G. Let $W' \subset W$ be the set of vertices of degree 6. A vertex $x \in W'$ must be the corner of exactly two rectangles that do not share a second corner. Hence x has two neighbors lying on a common line segment starting from x (see Figure 11). Denote the closer of these two neighbors by y. Observe that y must be the corner of exactly two rectangles. Now for every such pair x and y direct the edge xy in G from x to y. Note that this procedure will never direct y to x. Thus all vertices in W' have outdegree exactly one.

Let us first color the vertices of W' with indegree zero. Let $x \in W'$ be a vertex with indegree zero. The vertex x has outdegree one, so x has an uncolored neighbor, thus there is an available color for x. Color x with an available color and remove x from W'. Repeat this step until no vertex with indegree zero remains in W'. Now every vertex in W' has outdegree and indegree equal to 1. Therefore the remaining vertices of W' can be partitioned into directed cycles. The vertices on a directed cycle in W' have at most 4 previously colored neighbors, so each vertex has a list of at least two available colors.

If we examine these cycles in R it is clear that the edges must alternate between vertical and horizontal orientation. Thus these directed cycles are of even length. The list-chromatic number of an even cycle is 2, hence each cycle can be colored properly. Now all vertices in W' are colored. The remaining uncolored vertices in G are the vertices of $W - W'$. These vertices have degree at most 5 and thus all have an available color.

□

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