On linear forbidden submatrices

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Abstract

In this paper we study the extremal problem of finding how many 1 entries an n by n 0-1 matrix can have if it does not contain certain forbidden patterns as submatrices. We call the number of 1 entries of a 0-1 matrix its weight. The extremal function of a pattern is the maximum weight of an n by n 0-1 matrix that does not contain this pattern as a submatrix. We call a pattern (a 0-1 matrix) linear if its extremal function is $O(n)$. Our main results are modest steps towards the elusive goal of characterizing linear patterns. We find novel ways to generate new linear patterns from known ones and use this to prove the linearity of some patterns. We also find the first minimal non-linear pattern of weight above 4. We also propose an infinite sequence of patterns that we conjecture to be minimal non-linear but have $\Omega(n \log n)$ as their extremal function. We prove a weaker statement only, namely that there are infinitely many minimal not quasi-linear patterns among the submatrices of these matrices. For the definition of these terms see below.

1 Introduction

The extremal theory of 0-1 matrices with respect to forbidden submatrices was initiated by the papers [3, 1] more than 15 years ago. It has since attracted a lot of research. Applications to combinatorial geometry were present since the first papers, later in [7, 10] this theory was applied to solve the noted Stanley-Wilf conjecture of enumerative combinatorics. This extremal theory of matrices can be considered as a Turán type extremal theory of bipartite graphs with a linear order on the vertices. See more on this connection in [11] and see [2] on the related notion of convex geometric graphs.

1.1 Definitions

We start with the basic definitions. In this paper we consider 0-1 matrices. We consider 1 entries as representing "present" while 0 entries represent "missing". In keeping with this we call replacing a 1 entry by 0 in a matrix *deleting* that entry. We say that the 0-1 matrix A *represents* the same size matrix B if $B = A$ or B is obtained from A by deleting several 1's. We say that a 0-1 matrix A *contains* another 0-1 matrix B if a submatrix of A represents B. Notice that we do not allow the rows or columns to be permuted and therefore containment crucially depends on the order of the rows/columns. We say A *avoids* B if A does not contain B.

The *weight* of a 0-1 matrix P is the number of its 1 entries, denoted by $w(P)$. To avoid the trivial case of an all 0 matrix (contained in every matrix of appropriate size) we define a *pattern* to be a 0-1 matrix of weight at least 1. Our main interest is to find the *extremal function* $\mathrm{ex}(n, P)$ of the pattern P for specific patterns, where $\mathrm{ex}(n, P)$ is defined to be the maximal weight of an n by n 0-1 matrix avoiding P.

1.2 Linearity

We call a pattern P *linear* if $\mathrm{ex}(n, P) = O(n)$, otherwise P is *non-linear*. Characterizing linear patterns is of special interest but very little is known about them. Proving a conjecture of Füredi and Hajnal [4] Marcus and Tardos [10] show that permutation matrices are linear. By a result of Klazar and Valtr [9] on Davenport-Schinzel sequences certain *bitonic patterns* are also linear (see definition in Section 2 before Theorem 2.6). Beyond this only a few small patterns were shown to be linear and there were a few simple reduction rules in [4, 12] that implied the linearity of certain patterns if suitable submatrices were linear. In Section 2 we establish two new reductions and use them to prove linearity of certain patterns.

We call a pattern P *minimal non-linear* if it is non-linear but all patterns $Q \neq P$ contained by P are linear. Clearly, a pattern is linear if and only if it avoids all minimal non-linear patterns.

The order of magnitude of all patterns of weight at most four was established in [4, 12], so all linear and minimal non-linear patterns are known of weight at most four. However no minimal non-linear pattern has been known of larger weight and in fact finding such was raised in [12] as an open problem. In Section 3 we present a minimal non-linear pattern H_0 of weight 5. We establish that $\mathrm{ex}(n, H_0) = \Theta(n \log n)$. In fact, we give an infinite sequence of patterns H_i and we conjecture that each of them is minimal non-linear. We show that they are non-linear, moreover $\mathrm{ex}(n, H_i) = \Omega(n \log n)$ but we could not prove minimality or even that they contain infinitely many distinct minimal non-linear patterns. Instead we introduce *quasi-linearity*, a relaxation of linearity, see below, and prove a similar statement for that notion.

1.3 Quasi-linearity

We call a pattern *light* if it contains exactly one 1 entry in every column.

The close connection between the extremal function of light matrices and the Davenport-Schinzel theory of sequences was first noted in a special case by Füredi and Hajnal [4] and was developed later by Klazar. For us, the most important consequence of the connection is the following result of Klazar [7, 8].

Theorem 1.1. (Klazar [7, 8]) *For any light 0-1 matrix A there exists a constant c such that*

$$\mathrm{ex}(n, A) \leq n \cdot 2^{(\alpha(n))^c}.$$

Here α is the extremely slowly growing but unbounded inverse of Ackermann's function. As [8] is not easily accessible we include the simple deduction of this result from a fundamental result of [6] in Section 2.

The above result motivates that we call *quasi-linear* a function f if $f(n) \le n \cdot 2^{(\alpha(n))^c}$ for some c. We call a pattern P *quasi-linear* if $\mathrm{ex}(n,P)$ is quasi-linear. With this terminology Theorem 1.1 states that light patterns are quasi-linear. We call P *minimal not quasi-linear* if P is not quasi-linear but every pattern $Q \neq P$ that P contains is quasi-linear.

Our bounds on $\mathrm{ex}(n,H_i)$ show that the patterns H_i are not quasi-linear. Still short of proving that they are minimal not quasi-linear patterns in Section 3 we show that they contain infinitely many distinct minimal not quasi-linear patterns.

The results in this paper appeared in the Master's thesis of the author [5].

2 Reductions and connection to Davenport-Schinzel theory

In the paper [4] the systematic study of the extremal functions $\mathrm{ex}(n,P)$ was largely based on *reductions*: rules that determined the order of the magnitude of the extremal function $\mathrm{ex}(n,P)$ of a pattern P from that of a simpler pattern P'. In Lemma 2.3 of [12] these reductions and some new ones are collected. Here we state a a simple reduction from [12] and go on to state and prove two novel reductions. We also give an example of how the linearity of a pattern can be established using them.

Lemma 2.1. ([12]) *Let $A = (a_{i,j})$ be a k by l pattern and assume that for some indices $1 \le i_0 \le k$ and $1 \le j_0 \le l$ we have $a_{i_0,j_0} = a_{i_0,j_0+1} = 1$ and let $m \ge 1$ be an integer. Consider the k by $l+m$ pattern A' obtained from A by adding m new columns between the columns j_0 and j_0+1 of A. The new columns have a single 1 entry at row i_0. We have*

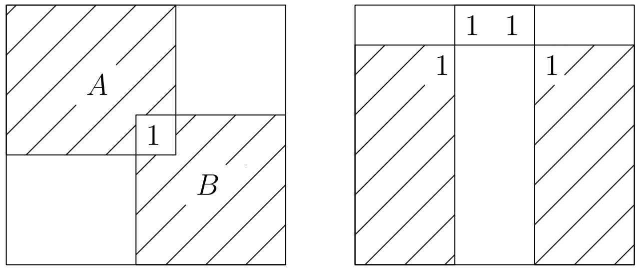
$$\mathrm{ex}(n,A') = \Theta(\mathrm{ex}(n,A)).$$

Proof. The result follows from the repeated application of the $m = 1$ special case which is stated as Lemma 2.3/g of [12]. □

Our first new reduction is very simple.

Theorem 2.2. *Assume the k_1 by l_1 pattern $A = (a_{i,j})$ has a 1 entry in its lower right corner (i.e., $a_{k_1,l_1} = 1$) and the k_2 by l_2 pattern $B = (b_{i,j})$ has a 1 entry in its upper left corner (i.e., $b_{1,1} = 1$). Let C be the pattern obtained by merging A and B at their mentioned corners, i.e., let $C = (c_{i,j})$ be the $k_1 + k_2 - 1$ by $l_1 + l_2 - 1$ pattern defined by $c_{i,j} = 1$ if and only if either $i \le k_1$, $j \le l_1$, and $a_{i,j} = 1$ or $i \ge k_1$, $j \ge l_1$, and $b_{i-k_1+1,j-l_1+1} = 1$. See Figure 1(a). We have*

$$\max(\mathrm{ex}(n,A),\mathrm{ex}(n,B)) \le \mathrm{ex}(n,C) \le \mathrm{ex}(n,A) + \mathrm{ex}(n,B).$$



(a) Construction of the matrix C in Theorem 2.2

(b) Construction of the matrix B from A (shaded) in Theorem 2.3

Figure 1: The two new reductions

Proof. The first inequality is trivial as both A and B are contained in C. For the second consider an n by n 0-1 matrix M avoiding C. We need to prove $w(M) \leq \mathrm{ex}(n, A) + \mathrm{ex}(n, B)$.

We say that a 1 entry in M is of type A if it is the lower right corner of a submatrix of M representing A. Consider the matrix M' obtained from M by deleting all 1 entries of type A, this matrix avoids A and therefore $w(M') \leq \mathrm{ex}(n, A)$. Similarly, we say that a 1 entry of M is of type B if it is the upper left corner of a submatrix representing B and notice that the matrix M'' obtained from M by deleting these entries avoids B and therefore $w(M'') \leq \mathrm{ex}(n, B)$. Finally notice that no 1 entry of M is both of type A and of type B as the submatrices proving these statements together would prove that M contains C. Therefore $w(M) \leq w(M') + w(M'') \leq \mathrm{ex}(n, A) + \mathrm{ex}(n, B)$ as needed. □

Our second reduction is as follows:

Theorem 2.3. *Let $A = (a_{i,j})$ be a k by l pattern with $a_{1,m} = a_{1,m+1} = 1$. We let B be the pattern obtained from A by adding a new first row containing two 1 entries between columns m and $m+1$ of A. See Figure 1(b). More precisely, $B = (b_{i,j})$ is a $k+1$ by $l+2$ pattern with $b_{i,j} = 1$ for $i > 1$ if and only if either $j \leq m$ and $a_{i-1,j} = 1$ or $j \geq m+3$ and $a_{i-1,j-2} = 1$ and $b_{1,j} = 1$ if and only if $j = m+1$ or $m+2$. We have*

$$\mathrm{ex}(n, B) = \Theta(\mathrm{ex}(n, A)).$$

The proof of this reduction is much more involved. It is based on the connection between the extremal functions studied here and Davenport-Schinzel theory. We start with a few definitions and results from this theory and give a short overview of the connection to the extremal function of light patterns. In the proof of this result we extend this connection to matrices that are not necessarily light.

For $k \geq 1$ we use the term *k-sequence* for a sequence of positive integers not exceeding k. The *length* $|s|$ of a sequence s is the number of its elements.

We call two appearances of the same value $s_j = s_k$ in the sequence $s = (s_i)$ an l-*repetition* if $1 \le |k-j| < l$. We call a 2-repetition an *immediate repetition.* A k-sequence $c = (c_i)$ *represents* an l-sequence $d = (d_i)$ of the same length if $c_i = f(d_i)$ for some injective function f and all i. We say that a sequence c *contains* another sequence d if a subsequence of c represents d.

Davenport-Schinzel theory estimates the maximum length $\mathrm{ex}(k,c)$ of a k-sequence without l-repetitions that does not contain the l-sequence c.

For an n by m 0-1 matrix $A = (a_{i,j})$ we define the *sequence of* A to be the n-sequence $s(A)$ obtained as the concatenation of m blocks such that the jth block consists of the integers i with $a_{i,j} = 1$ in increasing order.

Clearly, $|s(A)| = w(A)$ for any pattern A. Although for an n by m 0-1 matrix A the sequence $s(A)$ may contain immediate repetitions it is clear that we can get rid of all l-repetitions by deleting at most $(m-1)(l-1)$ entries (at most $l-1$ from each block, none from the first block).

It is easy to see that if the 0-1 matrix A contains the pattern B, then $s(A)$ contains $s(B)$. Unfortunately, the converse is not true in general. But it is "almost" true for light patterns. In fact, two previously mentioned statements are proved using this connection. They use the following two results on Davenport-Schinzel sequences:

Theorem 2.4. (Klazar [6]) *For any l-sequence u we have*

$$\mathrm{ex}(n,u) = n \cdot 2^{O((\alpha(n))^{|u|-4})}.$$

Theorem 2.5. (Klazar, Valtr [9]) *If the l-sequence u consists of an increasing sequence followed by a decreasing sequence followed by yet another increasing sequence, then*

$$\mathrm{ex}(n,u) = O(n).$$

The sequence $u = 11..122..2...ll..l(l-1)(l-1)...(l-1)...22...211...122...2...ll...l$ is a typical example for which Theorem 2.5 can be applied. We recall the proof of Theorem 1.1 using Theorem 2.4 (see [8]).

Proof of Theorem 1.1. Let k and l be positive integers and consider the l-sequence s obtained by repeating the sequence $1, 2, \ldots, l$ $2k$ times. It is easy to see that if $s(A)$ contains s for a 0-1 matrix A, then A contains all light l by k patterns P. Therefore considering the maximal weight n by n 0-1 matrix A that does not contain such a pattern P the n-sequence $s(A)$ does not contain s. After removing at most $(n-1)(l-1)$ elements from $s(A)$ it will be l-repetition free and will still not contain s. By Theorem 2.4 we have

$$\mathrm{ex}(n,P) = w(A) = |s(A)| \le \mathrm{ex}(n,s) + nl = n \cdot 2^{O((\alpha(n))^{2kl-4})}.$$

□

We call a light pattern P *bitonic* if $s(P)$ consists of an increasing segment followed by a decreasing segment. As we mentioned in the Introduction bitonic patterns are linear. This was known as a consequence of Theorem 2.5. Now we can say this is also a consequence of our Theorem 2.3 combined with Lemma 2.1 and the trivial observation that patterns contained in linear patterns are also linear.

Our proof of Theorem 2.3 can be considered as an adaptation of the proof of Theorem 2.5 to matrices. In particular we use the following definition and lemma from [9] (where it appears in a more general form).

Let $a = (a_i)$ be an l-sequence of length m. We call the index i_0 *covered* in a if there are indices $1 \leq j_1 \leq i_0 \leq j_2 \leq m$ such that the subsequence (interval) $(a_{j_1}, a_{j_1+1}, \ldots, a_{j_2})$ of a contains at most 16 occurrences of the value a_{i_0} and contains at least two occurrences of some integer $b < a_{i_0}$.

Lemma 2.6. ([9]) *Let s be an l-sequence without immediate repetitions. If $|s| > 1440l$ then there exist at least $|s|/10$ indices $1 \leq i \leq |s|$ that are covered in s.*

Proof of Theorem 2.3. Clearly $\mathrm{ex}(n, A) \leq \mathrm{ex}(n, B)$ as B contains A. We need to give an upper bound of $\mathrm{ex}(n, B)$.

Let $M = (m_{i,j})$ be an n by n 0-1 matrix of maximal weight avoiding B. Let $s = (s_i)$ be obtained from the n-sequence $s(M)$ by removing immediate repetitions. We have

$$|s| > |s(M)| - n = w(M) - n = \mathrm{ex}(n, B) - n.$$

If $|s| \leq 1440n$, then $\mathrm{ex}(n, B) < 1441n = O(\mathrm{ex}(n, A))$ since $\mathrm{ex}(n, A) \geq n$. Therefore we can and will assume that $|s| > 1440n$ and Lemma 2.6 applies. Let s' be the subsequence of s consisting of the elements s_i for which i is covered in s. By Lemma 2.6

$$|s'| \geq |s|/10.$$

Each element of the sequence $s(M)$ corresponds to a 1 entry in M. Using this correspondence the subsequence s' determines a subset of the 1 entries in M. Let M' be the matrix obtained from M by deleting all other 1 entries, keeping only the ones corresponding to the subsequence s'.

Let A' be the pattern obtained from A by adding 33 new columns between columns m and $m+1$ of A such that these columns have a single 1 entry in the first row. Recall that $a_{1,m} = a_{1,m+1} = 1$. We call the 1 entries in the inserted columns the new 1 entries. By Lemma 2.1 we have

$$\mathrm{ex}(n, A') = O(\mathrm{ex}(n, A)).$$

The main observation is that M' does not contain A'. Assume for contradiction that a submatrix M_1 of M' represents A'. If we delete the columns of the 1 entries of M_1 corresponding to the new 1 entries in A' we obtain a submatrix M_2 of M representing A. Each 1 entry in A' corresponds to a 1 entry in M_1 and therefore in M. In particular, the middle one of the 33 new 1 entries corresponds to some 1 entry in some row c of M. This 1 entry of M corresponds to a c in the sequence $s(M)$ that made it to the subsequences s and s'. Therefore the index corresponding to this element in s is covered in s. This means the existence of an interval of s containing this value c and at most 15 other occurrences of c and at least 2 appearances of a value $b < c$. These two appearances of b in s correspond to two 1 entries in M again. If we now add the row and the columns of these two 1 entries in M to the submatrix M_2 we obtain another submatrix M_3. We claim that M_3 represents B. Indeed, the row of the two extra 1 entries is the first row of M_3 as $b < c$ and the columns of these 1 entries must be between the columns of the 1 entries corresponding to the first and last new 1 entries in A' (inclusive) as otherwise the interval of s containing the two b entries would contain at least 17 of the c entries corresponding to the new 1 entries in A'. We obtain a contradiction here since M was supposed to avoid B. The contradiction proves that M' indeed avoids A', therefore

$$|s'| = w(M') \leq \text{ex}(n, A').$$

Combining the four displayed inequalities in this proof one gets $\text{ex}(n, B) = O(\text{ex}(n, A) + n)$. This finishes the proof of Theorem 2.3 since $\text{ex}(n, A) \geq n$. □

We remark that by Lemma 2.1 we can add any constant number of new columns between columns m and $m + 1$ of A and Theorem 2.3 still holds.

We use Theorem 2.3 or its generalization above to prove the linearity of certain patterns. We can start from the 1 by k all-1 pattern that is trivially linear. By repeated application of the above result we conclude that all bitonic patterns are linear. As we mentioned this has been known. It leads to new results however if we apply Theorem 2.3 to patterns that are not light (for the patterns mentioned below see the Appendix):

Corollary 2.7. *The patterns L_2 and L_3 are linear.*

Proof. For the linearity of pattern L_3 it is enough by Theorem 2.3 to prove the linearity of L_1. This is done in [12]. The pattern L_2 is contained in L_3 and so its linearity follows. □

3 On minimal non-linear and not quasi-linear patterns

First we define the infinite sequence of patterns H_k for $k \geq 0$ that we conjecture to be minimal non-linear. For $k \geq 0$ let $H_k = (h_{i,j})$ be the m by m pattern with $m = 3k + 4$ and with all entries zero except for the following ones:

$$h_{4,1} = h_{1,2} = h_{1,3} = h_{m-1,m} = h_{m-2,m} = 1,$$
$$h_{3l+4,3l+1} = h_{3l-1,3l+3} = h_{3l,3l+2} = 1 \qquad (1 \leq l \leq k).$$

Note that $w(H_k) = 3k + 5$ and H_k is symmetric around the diagonal from $h_{1,m}$ to $h_{m,1}$. See the Appendix for H_0 and H_1.

Unfortunately we can verify minimal non-linearity only for the first pattern in this sequence. H_0 is the only pattern of weight above 4 that is known to be minimal non-linear.

Theorem 3.1. *The pattern H_0 is minimal non-linear and we have $ex(n, H_0) = \Theta(n \log n)$.*

Proof. Recall that the order of magnitude of the extremal function of all patterns with weight at most 4 was found in [4, 12]. In particular, for all patterns $P \neq H_0$ contained in H_0 we have $\mathrm{ex}(n, P) = O(n)$. So the minimal non-linear property follows from the claimed result on $\mathrm{ex}(n, H_0)$.

For the lower bound we use the n by n 0-1 matrices $A_n = (a_{i,j})$ defined by $a_{i,j} = 1$ if and only if $j - i = 2^k$ for some integer k. We remark that A_n is symmetric around the diagonal from $h_{1,n}$ to $h_{n,1}$ and in [12] it is shown that A_n avoids Q_1 and Q_2 and its weight $w(A_n) = \sum_{k=0}^{\lfloor \log_2 n \rfloor}(n - 2^k) \geq n \log_2 n - n$ is within $O(n)$ to the maximal weight of any n by n 0-1 matrix avoiding either of those patterns. Here we need to prove that the matrix A_n avoids H_0. We prove that for $1 \leq i_1 \leq i_2 < i_3 \leq i_4 \leq m$ and $1 \leq j_1 \leq j_2 < j_3 \leq j_4 \leq m$ we don't have $a_{i_4,j_1} = a_{i_1,j_2} = a_{i_1,j_3} = a_{i_2,j_4} = a_{i_3,j_4} = 1$. This guarantees that A_n avoids a few patterns including Q_1, Q_2 and more importantly H_0.

Assume for contradiction that we have 1 in all the five positions mentioned above. Therefore we have $j_1 - i_4 > 0$ and $j_3 - i_1 = 2^{k_3}$, $j_2 - i_1 = 2^{k_2}$ for some integers $k_3 > k_2$ (as $j_3 > j_2$). Thus $i_3 - i_2 \leq i_4 - i_1 \leq j_2 - i_1 + i_4 - j_1 < j_2 - i_1 = 2^{k_2} \leq (j_3 - i_1) - (j_2 - i_1) = j_3 - j_2$. Symmetrical argument shows that $j_3 - j_2 < i_3 - i_2$. The contradiction proves our claim and with it the lower bound $\mathrm{ex}(n, H_0) \geq n \log n - n$.

For the upper bound we apply Theorem 2.3 for the pattern Q_2. We obtain a 4 by 5 weight 6 pattern Q_2' with $\mathrm{ex}(n, Q_2') = \Theta(\mathrm{ex}(n, Q_2)) = \Theta(n \log n)$ (we used that $\mathrm{ex}(n, Q_2)) = \Theta(n \log n)$ [12]). As Q_2' contains H_0 we also have $\mathrm{ex}(n, H_0) \leq \mathrm{ex}(n, Q_2') = \Theta(n \log n)$. ☐

We can generalize the above lower bound on $\mathrm{ex}(n, H_0)$ as follows:

Theorem 3.2. *For any $k \geq 0$ for the pattern H_k we have $ex(H_k, n) = \Omega(n \log n)$.*

It is tempting to use the same matrices A_n for this more general lower bound. For H_1 this approach works as a similar reasoning gives that A_n avoids it for any n. Unfortunately H_2 is contained in A_n for $n \geq 74$ and in fact for any $k \geq 2$ and large enough n the matrix A_n contains H_k. This is why we introduce a modified construction.

Let $B_n = (b_{i,j})$ be the n by n 0-1 matrix where $b_{i,j} = 1$ if and only if $j - i = 3^k$ for some integer k. The weight of B_n is $w(B_n) = \sum_{k=0}^{\lfloor \log_3 n \rfloor}(n - 3^k) \geq n \log_3 n - n$. Note that the 1 entries in B (just as in A) are arranged in diagonals (one for every k).

Lemma 3.3. *Assume that for the row indices $1 \leq i_1 \leq i_2 < i_3 < i_4 < i_5 \leq n$ and column indices $1 \leq j_1 < j_2 < j_3 < j_4 \leq j_5 \leq n$ in B_n we have $b_{i_1,j_3} = b_{i_2,j_2} = b_{i_3,j_5} = b_{i_4,j_4} = b_{i_5,j_1} = 1$. We have $j_3 - j_2 - i_2 + i_1 < j_5 - j_4 - i_4 + i_3$.*

Proof. By the assumption and the definition of B_n we have $j_3 - i_1 = 3^{k_1}$ and $j_2 - i_2 = 3^{k_2}$ for some positive integers $k_1 > k_2$ (as $j_3 - i_1 > j_2 - i_2$). Similarly

$j_5 - i_3 = 3^{l_1}$ and $j_4 - i_4 = 3^{l_2}$ for some positive integers $l_1 > l_2$ (as $j_5 - i_3 > j_4 - i_4$). Finally, $j_1 - i_5 = 3^{k_3} \geq 1$ for some positive integer k_3.

As $(j_5 - j_4) + (i_4 - i_3) = (j_5 - i_3) - (j_4 - i_4) = 3^{l_1} - 3^{l_2} \geq 2 \cdot 3^{l_2}$ and $j_3 - j_2 < (j_4 - j_1) + (i_5 - i_4) = (j_4 - i_4) - (j_1 - i_5) < 3^{l_2}$ we have $(j_5 - j_4) + (i_4 - i_3) > 2(j_3 - j_2)$.

By symmetry we also have $(j_3 - j_2) + (i_2 - i_1) > 2(i_4 - i_3)$. Summing these two inequalities yields the claim. □

Proof of Theorem 3.2. It is enough to prove that B_n avoids H_k for any $n > 0$ and $k \geq 0$ as we have seen that $w(B(n)) = \Theta(n \log n)$. Assume for contradiction that B_n contains H_k. Take a submatrix of B_n representing H_k. Let its row and column indices be $i_1 < i_2 < \ldots < i_m$ and $j_1 < j_2 < \ldots < j_m$ where $m = 3k + 4$. We set $i_{-1} = i_0 = i_1$ and $j_{m+2} = j_{m+1} = j_m$. For $0 \leq l \leq k + 1$ let $x_l = j_{3l+3} - j_{3l+2} - i_{3l} + i_{3l-1}$. Let $0 \leq l \leq k$ and let us use Lemma 3.3 for the row indices $i_{3l-1} \leq i_{3l} < i_{3l+2} < i_{3l+3} < i_{3l+4}$ and the column indices $j_{3l+1} < j_{3l+2} < j_{3l+3} < j_{3l+5} \leq j_{3l+6}$. As the submatrix represents H_k the 1 entries needed for the lemma to apply are present in B_n and we obtain $x_l < x_{l+1}$. This contradicts the fact that $x_0 = j_3 - j_2 > 0$ and $x_{k+1} = i_{m-2} - i_{m-1} < 0$. the contradiction proves that B_n does not contain H_k and finishes the proof of the theorem. □

As mentioned earlier, we conjecture that the patterns H_k are minimal non-linear patterns. The above theorem guarantees that these patterns are non-linear. As a consequence each contains a minimal non-linear pattern. Unfortunately we cannot rule out that all H_k for $k \geq 1$ contain the *same* minimal non-linear pattern. The pattern G_1 (see Appendix) is contained in each of them and although it seems to be linear this remains an open problem. But surely G_1 is quasi-linear by Theorem 1.1, while H_k is not quasi-linear by Theorem 3.2. A weaker conjecture claims that the patterns H_k are minimal not quasi-linear patterns. To prove this conjecture we would need to prove that by deleting any 1 entry from H_k for any k we get a quasi-linear pattern. We cannot prove this either, but we can prove this for enough of the 1 entries in H_k to conclude that the patterns H_k contain infinitely many distinct minimal not quasi-linear patterns.

We call a 1 entry of a pattern *important* if deleting that entry yields a quasi-linear pattern.

Lemma 3.4. *For any $k \geq 0$ there are at least $k + 5$ important 1 entries in H_k.*

Proof. Deleting either of the two 1 entries in the last column of H_k yields a light matrix. Therefore by Theorem 1.1 these two entries are important. By symmetry the two 1 entries in the first row are also important.

We claim that the 1 entries in position $((3l + 4), (3l + 1))$ for $0 \leq l \leq k$ are also important. This gives up a total of $k + 5$ important entries as claimed.

To see the claim let us fix $0 \leq l \leq k$ and let P be the pattern obtained from H_k by deleting the entry in position $((3l + 4), (3l + 1))$. We need to prove that P is quasi-linear. Let A' be the submatrix formed by the first $3l + 1$ rows and the first $3l+3$ columns of P. Let B' be the submatrix of P formed by the remaining rows and the remaining columns. Notice that P has no 1 entry outside these two submatrices, so P can be obtained from A' and B' arranging them diagonally.

Notice that diagonally arranging two linear patterns can yield a pattern with extremal function $\theta(n \log n)$ as (a rotated copy of) Q_2 shows. But Theorem 2.2 shows that such an increase in the extremal function does not happen in certain cases.

Let us obtain A by adding a new last row and a new last column to A' and inserting a single new 1 entry in their intersection (in other words A is obtained by diagonally arranging A' and the trivial 1 by 1 pattern of a single 1 entry). Similarly, let us obtain B by adding a new first row and a new first column to B' with a single new 1 entry in their intersection. We can now apply Theorem 2.2 to A and B to obtain a pattern C. C is actually obtained by diagonally arranging A', the trivial pattern, and B'. As P is contained in C we have $\mathrm{ex}(n,P) \leq \mathrm{ex}(n,C)$. By the theorem we have $\mathrm{ex}(n,C) \leq \mathrm{ex}(n,A) + \mathrm{ex}(n,B)$. To prove that P is quasi-linear it is therefore enough to prove that both A and B are quasi-linear.

We prove the quasi-linearity of A. The quasi-linearity of B follows by symmetry. The quasi-linearity of A follows from Theorem 1.1. Column $3l+1$ of A contains no 1 entry, all other columns contain a single 1 entry. Therefore A is contained in a light matrix A_1 (simply add a 1 entry in column $3l+1$) and therefore $\mathrm{ex}(n,A) \leq \mathrm{ex}(n,A_1)$ and the latter is quasi-linear by Theorem 1.1. Alternatively one can argue that the pattern obtained by deleting the empty column in A is quasi-linear and deleting empty columns does not alter the order of the magnitude of any pattern of weight at least 2. This finishes the proof of the lemma. □

Theorem 3.5. *There exist infinitely many pairwise distinct minimal not quasi-linear patterns contained in the patterns H_k.*

Proof. Theorem 3.2 states that H_k is not quasi-linear. Therefore H_k must contain a minimal not quasi-linear pattern. We obtain such a pattern H'_k by deleting 1 entries and empty rows and columns from H_k in a way that does not cause the remaining pattern to be quasi-linear. Clearly, important 1 entries of H_k cannot be deleted, so by Lemma 3.4 we have $w(H'_k) \geq k+5$. Thus the weight of H'_k is unbounded, so there must be infinitely many distinct patterns among them. □

We remark that a similar argument to the one in the proof of Lemma 3.4 shows that deleting the two 1 entries in H_k at positions $(3l-1, 3l+2)$ and $(3l, 3l+1)$ for some $1 \leq l \leq k$ also yields a quasi-linear pattern. This can be used to prove $w(H'_k) \geq 2k+5$. One can also show that all the patterns H'_k are pairwise distinct but showing that deleting just one of these two 1 entries also yields to (quasi-)linear patterns seems to be harder.

4 Conjectures

If we want to use the proof of Theorem 3.5 to obtain infinitely many minimal non-linear patterns, then we need that the patterns A and B in Lemma 3.4 have linear extremal function. Note that the shape of A and B is symmetrical, thus it is enough to prove this for A. To make it more precise, let G_k be the matrix obtained from H_k by deleting the column containing the 1 entry in the last row, the last column and the last three rows (see the Appendix for G_1). Clearly, any A which can appear in the above proof is contained in a G_k for some k. Thus, if $ex(n, G_k) = O(n)$ would be true for every k then the proof would give that the patterns H_k reduce to infinitely many pairwise different minimal non-linear patterns.

In section 1.2 we mentioned that the patterns with weight at most 4 are classified. Though, there are some patterns with weight 5 whose extremal function is not determined yet. At the end of section 2 we proved that L_2 is linear. In the previous section we proved that H_0 has extremal function $\Theta(n \log n)$. For the weight 5 pattern G_1 the extremal function is not determined yet.

Conjecture 4.1.

1. *For the pattern G_1 we have $ex(n, G_1) = O(n)$.*
2. *For the pattern G_k we have $ex(n, G_k) = O(n)$ $(k \geq 1)$.*

As already mentioned in the beginning of Section 3, the patterns H_k are not only prime candidates for containing infinitely many non-linear patterns, but the patterns H_k can be minimal non-linear patterns themselves:

Conjecture 4.2.

1. *There are infinitely many minimal non-linear patterns.*
2. *The patterns H_k are minimal non-linear patterns.*

Note that Conjecture 4.1 would prove the first statement of this conjecture.

Notice that the patterns G_k can be obtained from a permutation pattern by doubling the column containing the 1 entry in its first row. As already mentioned in Section 1.2 permutation patterns have linear extremal function [10]. It may be true that by doubling one of its columns the extremal function remains linear. A weaker claim would be enough, namely that by doubling the column containing the 1 entry in its first row the extremal function remains linear. Note that these are not true for arbitrary patterns, as H_0 can be obtained from a linear pattern by doubling the column containing the 1 entry in its first row (the linearity of the pattern obtained from H_0 by deleting its second column follows easily from the linearity of L_1 using the reductions presented in [12]), yet its extremal function is $\Theta(n \log n)$. Besides, it is also necessary to put the new column next to the one which was doubled. Indeed, the matrix S_2 can be obtained from a permutation pattern by adding the copy of the column containing the 1 entry in the first row after the existing columns, though $ex(S_2, n) = \Theta(n\alpha(n))$ [4]. For permutation patterns even the stronger claim, that we can double all columns without increasing the order of magnitude, may be true.

Conjecture 4.3.

1. *For any permutation pattern by doubling the column containing the 1 entry in its first row we obtain a pattern with linear extremal function.*
2. *By doubling one column of a permutation pattern we obtain a pattern with linear extremal function.*
3. *By doubling every column of a permutation pattern we obtain a pattern with linear extremal function.*

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5 Appendix

List of patterns

In the table we use dots for the 1 entries and blank spaces for the 0 entries.

$$Q_1 = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \bullet \end{pmatrix} \qquad Q_2 = \begin{pmatrix} & \bullet & \bullet \\ & & \bullet \\ \bullet & & \end{pmatrix}$$

$$S_2 = \begin{pmatrix} & \bullet & & \bullet \\ & & \bullet & \\ \bullet & & & \end{pmatrix} \qquad L_1 = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \\ & & \bullet \end{pmatrix}$$

$$L_2 = \begin{pmatrix} & \bullet & \bullet & \\ & & & \bullet \\ \bullet & & & \\ & & & \bullet \end{pmatrix} \qquad L_3 = \begin{pmatrix} & & \bullet & \bullet & \\ & \bullet & & & \bullet \\ \bullet & & & & \\ & & & & \bullet \end{pmatrix}$$

$$H_0 = \begin{pmatrix} & \bullet & \bullet & \\ & & & \bullet \\ & & & \bullet \\ \bullet & & & \end{pmatrix} \qquad H_1 = \begin{pmatrix} & \bullet & \bullet & & & & \\ & & & & & \bullet & \\ & & & & \bullet & & \\ \bullet & & & & & & \\ & & & & & & \bullet \\ & & & & & & \bullet \\ & & & \bullet & & & \end{pmatrix}$$

$$G_1 = \begin{pmatrix} & \bullet & \bullet & & \\ & & & & \bullet \\ & & & \bullet & \\ \bullet & & & & \end{pmatrix}$$

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