Cross-Sperner families

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Abstract

A pair of families $(\mathcal{F},\mathcal{G})$ is said to be *cross-Sperner* if there exists no pair of sets $F\in\mathcal{F}, G\in\mathcal{G}$ with $F\subseteq G$ or $G\subseteq F$. There are two ways to measure the size of the pair $(\mathcal{F},\mathcal{G})$: with the sum $|\mathcal{F}|+|\mathcal{G}|$ or with the product $|\mathcal{F}|\cdot|\mathcal{G}|$. We show that if $\mathcal{F},\mathcal{G}\subseteq 2^{[n]}$, then $|\mathcal{F}||\mathcal{G}|\le 2^{2n-4}$ and $|\mathcal{F}|+|\mathcal{G}|$ is maximal if $\mathcal{F}$ or $\mathcal{G}$ consists of exactly one set of size $\lceil n/2\rceil$ provided the size of the ground set n is large enough and both $\mathcal{F}$ and $\mathcal{G}$ are non-empty.

1 Introduction

We use standard notation: $[n]$ denotes the set of the first n positive integers, 2^S denotes the power set of the set S and $\binom{S}{k}$ denotes the set of all k-element subsets of S. The complement of a set F is denoted by $\overline{F}$ and for a family $\mathcal{F}$ we write $\overline{\mathcal{F}}=\{\overline{F}:F\in\mathcal{F}\}$.

One of the first theorems in the area of extremal set families is that of Sperner [15], stating that if we consider a family $\mathcal{F}\subseteq 2^{[n]}$ such that no set $F\in\mathcal{F}$ can contain any other $F'\in\mathcal{F}$, then the number of sets in $\mathcal{F}$ is at most $\binom{n}{\lfloor n/2\rfloor}$ and equality holds if and only if $\mathcal{F}=\binom{[n]}{\lfloor n/2\rfloor}$ or $\mathcal{F}=\binom{[n]}{\lceil n/2\rceil}$. Families satisfying the assumption of Sperner's theorem are called *Sperner families* or *antichains*. The celebrated theorem of Erdős,

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Ko and Rado [6] asserts that if for a family $\mathcal{G} \subseteq \binom{[n]}{k}$ we have $G \cap G' \neq \emptyset$ for all $G, G' \in \mathcal{G}$ (families with this property are called *intersecting*), then the size of $\mathcal{G}$ is at most $\binom{n-1}{k-1}$ provided $2k \leq n$.

There have been many generalizations and extensions both to the theorem of Sperner and to the result by Erdős, Ko and Rado (two excellent but not really recent surveys are [4] and [5]). One such generalization is the following: a pair $(\mathcal{F}, \mathcal{G})$ of families is said to be *cross-intersecting* if for any $F \in \mathcal{F}, G \in \mathcal{G}$ we have $F \cap G \neq \emptyset$. Cross-intersecting pairs of families have been investigated for quite a while and attracted the attention of many researchers [2, 3, 7, 8, 9, 10, 11, 12]. The present paper deals with the analogous generalization of Sperner families that has not been considered in the literature. A pair $(\mathcal{F}, \mathcal{G})$ of families is said to be *cross-Sperner* if there exists no pair of sets $F \in \mathcal{F}, G \in \mathcal{G}$ with $F \subseteq G$ or $G \subseteq F$. There are two ways to measure the size of the pair $(\mathcal{F}, \mathcal{G})$: either with the sum $|\mathcal{F}| + |\mathcal{G}|$ or with the product $|\mathcal{F}| \cdot |\mathcal{G}|$. We will address both problems.

Clearly, $|\mathcal{F}| + |\mathcal{G}| \leq 2^n$ as by definition $\mathcal{F} \cap \mathcal{G} = \emptyset$. The sum 2^n can be obtained by putting $\mathcal{F} = \emptyset, \mathcal{G} = 2^{[n]}$. Thus, when considering the problem of maximizing $|\mathcal{F}| + |\mathcal{G}|$ we will assume that both $\mathcal{F}$ and $\mathcal{G}$ are non-empty.

We can reformulate our problem in a rather interesting way. Let $\Gamma_n = (V_n, E_n)$ be the graph with vertex set $V_n = 2^{[n]}$ and edge set $E_n = \{(F, G) : F, G \in V_n \;\; F \subsetneq G$ or $G \subsetneq F\}$. Then $\max\{|\mathcal{F}| + |\mathcal{G}|\} = 2^n - c(\Gamma_n)$, where $c(\Gamma_n)$ denotes the vertex connectivity of Γ_n. Moreover, if we let

$$F(n, m) = \max\{|\mathcal{G}| : \mathcal{G} \subseteq 2^{[n]}, \exists \mathcal{F} \subseteq 2^{[n]} \text{ with } |\mathcal{F}| = m, (\mathcal{F}, \mathcal{G}) \text{ is cross-Sperner}\},$$

then, denoting by $N_{\Gamma_n}(U)$ the neighborhood of U in Γ_n, we have

$$F(n, m) = 2^n - m - \min\{|N_{\Gamma_n}(\mathcal{F})| : \mathcal{F} \subseteq V_n, |\mathcal{F}| = m\}.$$

Thus determining $F(n, m)$ is equivalent to the isoperimetric problem for the graph Γ_n.

Let us mention that the cross-Sperner property of the pair $(\mathcal{F}, \mathcal{G})$ is equivalent to $(\mathcal{F}, \overline{\mathcal{G}})$ being cross-intersecting and cross-co-intersecting, i.e. for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ we have $F \cap \overline{G} \neq \emptyset$ and $F \cup \overline{G} \neq [n]$.

The rest of the paper is organized as follows. In Section 2, we consider the problem of maximizing $|\mathcal{F}| + |\mathcal{G}|$ and prove the following theorem.

Theorem 1.1. *There exists an integer n_0 such that if $n \geq n_0$ and the pair $(\mathcal{F}, \mathcal{G})$ is cross-Sperner with $\emptyset \neq \mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$, then*

$$|\mathcal{F}| + |\mathcal{G}| \leq F(n, 1) + 1 = 2^n - 2^{\lceil n/2 \rceil} - 2^{\lfloor n/2 \rfloor} + 2,$$

and equality holds if and only if $\mathcal{F}$ or $\mathcal{G}$ consists of exactly one set S of size $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ and the other family consists of all subsets of $[n]$ not contained in S and not containing S.

In Section 3, we address the problem of maximizing $|\mathcal{F}| \cdot |\mathcal{G}|$. Our result is the following theorem.

Theorem 1.2. *If $n \geq 2$ and $(\mathcal{F}, \mathcal{G})$ is cross-Sperner with $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$, then the following inequality holds:*

$$|\mathcal{F}||\mathcal{G}| \leq 2^{2n-4}.$$

This bound is best possible as shown by $\mathcal{F} = \{F \in 2^{[n]} : 1 \in F, n \notin F\}, \mathcal{G} = \{G \in 2^{[n]} : n \in G, 1 \notin G\}$.

Finally, Section 4 contains some concluding remarks and open problems.

2 Proof of Theorem 1.1

Before we start the proof of Theorem 1.1, let us introduce some notation and state a theorem that we will use in our proof. For a k-uniform family $\mathcal{F} \subseteq \binom{[n]}{k}$ let $\Delta\mathcal{F} = \{G \in \binom{[n]}{k-1} : \exists F \in \mathcal{F}, G \subset F\}$ be the *shadow* of $\mathcal{F}$. The following version of the shadow theorem is due to Lovász [13].

Theorem 2.1. [Lovász [13]] *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and let us define the real number x by $|\mathcal{F}| = \binom{x}{k}$. Then we have $\Delta\mathcal{F} \geq \binom{x}{k-1}$.*

For any $F \in 2^{[n]}$ we have $N_{\Gamma_n}(F) = 2^{|F|} + 2^{n-|F|} - 2$ which is minimized if $|F| = \lceil n/2 \rceil$. This proves $F(n,1) = 2^n - 2^{\lceil n/2 \rceil} - 2^{\lfloor n/2 \rfloor} + 1$ as stated in Theorem 1.1.

Proposition 2.2. *If a pair $(\mathcal{F}, \mathcal{G})$ maximizes $|\mathcal{F}| + |\mathcal{G}|$, then both $\mathcal{F}$ and $\mathcal{G}$ are convex families i.e. $F_1 \subset F \subset F_2$, $F_1, F_2 \in \mathcal{F}$ implies $F \in \mathcal{F}$.*

Proof. If F, F_1, F_2 are as above, then F can be added to $\mathcal{F}$ since any set containing F contains F_1 and any subset of F is a subset of F_2. □

Let $(\mathcal{F}, \mathcal{G})$ be a pair of cross-Sperner families and let F_0 and G_0 be sets of minimum size in $\mathcal{F}$ and $\mathcal{G}$.

Proposition 2.3. *If $|F_0| + |G_0| < \lceil n/2 \rceil - 1$, then $|\mathcal{F}| + |\mathcal{G}| < F(n,1)$.*

Proof. No set containing $F_0 \cup G_0$ can be a member of $\mathcal{F}$ or $\mathcal{G}$. □

As $(\mathcal{F}, \mathcal{G})$ is cross-Sperner if and only if $(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ is cross-Sperner, by taking complements (if necessary) and Proposition 2.3 we may and will assume that $m := |F_0| \geq \lfloor n/4 \rfloor$. Let us write $\mathcal{F}^* = \{F \in \mathcal{F} : F_0 \subsetneq F\}$. Subsets of F_0 are not in $\mathcal{F}$ by the minimality of F_0 and by the cross-Sperner property they cannot be in $\mathcal{G}$ either, thus to prove Theorem 1.1 we need to show that there exist more than $|\mathcal{F}^*|$ many sets that are not contained in $\mathcal{F} \cup \mathcal{G}$ and are not subsets of F_0. For any $F^* \in \mathcal{F}^*$ let us define

$$B(F^*) = \{F^* \setminus F_0' : F_0' \subseteq F_0, |F^* \setminus F_0'| < m\}.$$

Clearly, for any $F_1^*, F_2^* \in \mathcal{F}^*$ we have $B(F_1^*) \cap B(F_2^*) = \emptyset$ as they already differ outside F_0. By definition, no set in $\mathcal{B} := \cup_{F^* \in \mathcal{F}^*} B(F^*)$ is a subset of F_0. We have $\mathcal{B} \cap \mathcal{F} = \emptyset$ as all sets in $\mathcal{B}$ have size smaller than m and $\mathcal{B} \cap \mathcal{G} = \emptyset$ by the cross-Sperner property. Thus to prove Theorem 1.1 it is enough to show that $|\mathcal{F}^*| < |\mathcal{B}|$.

Note the following three things:

- $|B(F^*)| = \sum_{i=|F^* \setminus F_0|+1}^{m} \binom{m}{i}$,
- $\mathcal{F}^{**} = \{F^* \setminus F_0 : F^* \in \mathcal{F}^*\}$ is downward closed as $\mathcal{F}$ and $\mathcal{F}^*$ are convex,
- $|\mathcal{F}^{**}| = |\mathcal{F}^*|$.

Therefore the following lemma finishes the proof of Theorem 1.1 by choosing $\mathcal{A} = \mathcal{F}^{**}$, $k = m$ and $n' = n - |F_0|$.

Lemma 2.4. *Let $\emptyset \neq \mathcal{A} \subseteq 2^{[n']}$ be a downward closed family and $k \geq n'/3$. Then if n' is large enough, the following holds*

$$|\mathcal{A}| < \sum_{A \in \mathcal{A}} \sum_{i=|A|+1}^{k} \binom{k}{i}. \tag{1}$$

Proof. Let $a_i = |\{A \in \mathcal{A} : |A| = i\}|$ and $w(j) = \sum_{i=j+1}^{k} \binom{k}{i}$. Then we can formulate (1) in the following way:

$$\sum_{j=0}^{n'} a_j < \sum_{j=0}^{n'} a_j w(j). \tag{2}$$

Let x be defined by $a_{k-1} = \binom{x}{k-1}$. By Theorem 2.1 if $j \leq k-1$ then $a_j \geq \binom{x}{j}$. If we replace a_j by $\binom{x}{j}$ in (2), then the LHS decreases by $a_j - \binom{x}{j}$ and the RHS decreases by $(a_j - \binom{x}{j})w(j)$, which is larger. If $j \geq k-1$, then $a_j \leq \binom{x}{j}$ again by Theorem 2.1. If we replace a_j by $\binom{x}{j}$ in (2), then the LHS increases while the RHS does not change (as for $j \geq k$ we have $w(j) = 0$). Hence it is enough to prove

$$\sum_{j=0}^{n'} \binom{x}{j} < \sum_{j=0}^{n'} \binom{x}{j} w(j). \tag{3}$$

First we prove (3) for $x = n'$. In this case the LHS is $2^{n'}$ while the RHS is monotone increasing in k, thus it is enough to prove for $k = \lceil n/3 \rceil$. We will estimate the RHS from below by considering only one term of the sum. Clearly, $\binom{n'}{j} w(j) \geq \binom{n'}{j}\binom{k}{j+1} \geq \binom{n'}{j}\binom{n'/3}{j+1}$. Let us write $j = \alpha n'$ for some $0 \leq \alpha \leq 1/3$. Then by Stirling's formula we obtain

$$\binom{n'}{j}\binom{n'/3}{j+1} = \binom{n'}{\alpha n'}\binom{n'/3}{\alpha n'+1} = \Theta\left(\frac{1}{n'}\left(\frac{1}{\alpha^{2\alpha}(1-\alpha)^{1-\alpha}3^{1/3}(1/3-\alpha)^{1/3-\alpha}}\right)^{n'}\right).$$

The value of the fraction in parenthesis is larger than 2 for, say, $\alpha = 2/9$, thus (3) holds if n' is large enough and $x = n'$.

To prove (3) for arbitrary x, let $c = \binom{x}{k-1}/\binom{n'}{k-1}$. If $j > k-1$, then $c > \binom{x}{j}/\binom{n'}{j}$, while if $j < k-1$, then $c < \binom{x}{j}/\binom{n'}{j}$. By the $x = n'$ case we know

$$\sum_{j=0}^{n'} c\binom{n'}{j} < \sum_{j=0}^{n'} c\binom{n'}{j} w(j). \tag{4}$$

Let us replace $c\binom{n'}{j}$ by $\binom{x}{j}$ in this inequality. If $j > k-1$, then the LHS decreases and the RHS does not change. If $j = k-1$ none of the sides change by definition of c. If $j < k-1$, both sides increase, and the RHS increases more as $w(j) \geq 1$ for all $0 \leq j \leq k-1$. Hence the inequality holds and gives back (3), which finishes the proof of the lemma. □

We believe that Theorem 1.1 is valid for all n, but unfortunately Lemma 2.4 fails for small values of n.

3 Proof of Theorem 1.2

In this section we prove Theorem 1.2. Our main tool will be the following special case of the Four Functions Theorem of Ahlswede and Daykin [1]. To state their result for any pair $\mathcal{A}, \mathcal{B}$ of families let us write $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

Theorem 3.1. [Ahlswede-Daykin, [1]] *For any pair $\mathcal{A}, \mathcal{B}$ of families we have*

$$|\mathcal{A}||\mathcal{B}| \leq |\mathcal{A} \wedge \mathcal{B}||\mathcal{A} \vee \mathcal{B}|.$$

To prove Theorem 1.2 we will need the following lemma.

Lemma 3.2. *If $(\mathcal{F}, \mathcal{G})$ is a pair of cross-Sperner families, then the families $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F} \wedge \mathcal{G}$ and $\mathcal{F} \vee \mathcal{G}$ are pairwise disjoint.*

Proof. $\mathcal{F}$ and $\mathcal{G}$ are disjoint as some set $F \in \mathcal{F} \cap \mathcal{G}$ is a subset of itself and thus contradicts the cross-Sperner property. $\mathcal{F}$ and $\mathcal{G}$ are both disjoint from $\mathcal{F} \wedge \mathcal{G}$ and $\mathcal{F} \vee \mathcal{G}$ as $F \cap G \subseteq F, G$ and $F, G \subseteq F \cup G$. Finally, $\mathcal{F} \wedge \mathcal{G}$ and $\mathcal{F} \vee \mathcal{G}$ are disjoint as $F_1 \cap G_1 = F_2 \cup G_2$ would imply $F_1 \subseteq G_2$. □

Now we are able to prove Theorem 1.2.

Proof. Let $(\mathcal{F}, \mathcal{G})$ be a cross-Sperner pair of families. Clearly, if $|\mathcal{F}| + |\mathcal{G}| \le 2^{n-1}$, then the statement of the theorem holds. But if $|\mathcal{F}| + |\mathcal{G}| > 2^{n-1}$, then by Lemma 3.2 we have $|\mathcal{F} \wedge \mathcal{G}| + |\mathcal{F} \vee \mathcal{G}| < 2^{n-1}$ and thus by Theorem 3.1 we obtain $|\mathcal{F}||\mathcal{G}| \le |\mathcal{F} \wedge \mathcal{G}||\mathcal{F} \vee \mathcal{G}| \le 2^{2n-4}$. □

Corollary 3.3. *For $n \ge 2$, we have $F(n, 2^{n-2}) = 2^{n-2}$.*

4 Concluding remarks and open problems

One might wonder whether it changes the situation if we allow sets to belong to both $\mathcal{F}$ and $\mathcal{G}$ and we modify the definition of cross-Sperner families so that only pairs $F \in \mathcal{F}, G \in \mathcal{G}$ with $F \subsetneq G$ or $G \subsetneq F$ are forbidden. It is easy to see that the situation is the same when considering $|\mathcal{F}| + |\mathcal{G}|$. To prove that $|\mathcal{F}| + |\mathcal{G}| \le 2^n$ let us write $\mathcal{C} = \mathcal{F} \cap \mathcal{G}$ and if it is not empty, then $D(\mathcal{C}) := \{C \setminus C' : C, C' \in \mathcal{C}\}$ is disjoint both from $\mathcal{F}$ and $\mathcal{G}$ and a result by Marica and Schönheim [14] tells us that $|D(\mathcal{C})| \ge |\mathcal{C}|$. Note that the proof of Theorem 1.1 works in this case as well giving the upper bound $|\mathcal{F}| + |\mathcal{G}| \le F(n, 1) + 2$.

Although $F(n, m)$ is not known for most values, it is natural to generalize the problem to k-tuples of families: $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k$ is said to be cross-Sperner if for any $1 \le i < j \le k$ there is no pair $F \in \mathcal{F}_i$ and $F' \in \mathcal{F}_j$ with $F \subseteq F'$ or $F' \subseteq F$. One can consider the problems of maximizing $\sum_{i=1}^k |\mathcal{F}_i|$ and $\prod_{i=1}^k |\mathcal{F}_i|$. In the former case we need the extra assumption that all $\mathcal{F}_i$ are non-empty as otherwise the trivial upper bound 2^n is tight.

When maximizing the sum, it is natural to conjecture that in the best possible construction all but one family consists of one single set. By the cross-Sperner property, these sets together must form a Sperner family, therefore it might turn out to be useful to introduce

$$F^*(n, m) = \max\{|\mathcal{G}| : \mathcal{G} \subseteq 2^{[n]}, \exists \mathcal{F} \subseteq 2^{[n]}$$

$$\text{with } |\mathcal{F}| = m, (\mathcal{F}, \mathcal{G}) \text{ is cross-Sperner}, \mathcal{F} \text{ is Sperner}\}.$$

Problem 4.1. *Under what conditions is it true that if $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k$ form a k-tuple of non-empty cross-Sperner families, then*

$$\sum_{i=1}^{k} |\mathcal{F}_i| \leq k - 1 + F^*(n, k-1)?$$

Concerning maximizing the product of the $|\mathcal{F}_i|$, by Theorem 1.2 one obtains that

$$\prod_{i=1}^{k} |\mathcal{F}_i| = \left(\prod_{1 \leq i < j \leq k} |\mathcal{F}_i||\mathcal{F}_j| \right)^{\frac{1}{k-1}} \leq 2^{kn-2k}.$$

We conjecture that the following construction is optimal: let $l = l(k)$ be the smallest positive integer so that $k \leq \binom{l}{\lfloor l/2 \rfloor}$. Then there exists a Sperner family $\mathcal{S} = \{S_1, ..., S_k\} \subseteq 2^{[l]}$ of size k. Put $\mathcal{F}_i = \{F \subseteq [n] : F \cap [l] = S_i\}$. Clearly, the $\mathcal{F}_i$ form a k-tuple of cross-Sperner families and we have $\prod_{i=1}^{k} |\mathcal{F}_i| = 2^{k(n-l)}$. Unfortunately, already for $l = 3$ there is a gap of a factor of 8 between the upper bound and the size of our construction.

Conjecture 4.2. *If $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k \subseteq 2^{[n]}$ form a k-tuple of cross-Sperner families, then*

$$\prod_{i=1}^{k} |\mathcal{F}_i| \leq 2^{k(n-l)},$$

where l is the least positive integer with $\binom{l}{\lfloor l/2 \rfloor} \geq k$.

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