

Two-part set systems

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Abstract

The two part Sperner theorem of Katona and Kleitman states that if X is an n -element set with partition $X_1 \cup X_2$, and \mathcal{F} is a family of subsets of X such that no two sets $A, B \in \mathcal{F}$ satisfy $A \subset B$ (or $B \subset A$) and $A \cap X_i = B \cap X_i$ for some i , then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. We consider variations of this problem by replacing the Sperner property with the intersection property and considering families that satisfy various combinations of these properties on one or both parts X_1, X_2 . Along the way, we prove the following new result which may be of independent interest: let \mathcal{F}, \mathcal{G} be families of subsets of an n -element set such that \mathcal{F} and \mathcal{G} are both intersecting and cross-Sperner, meaning that if $A \in \mathcal{F}$ and $B \in \mathcal{G}$, then $A \not\subset B$ and $B \not\subset A$. Then $|\mathcal{F}| + |\mathcal{G}| < 2^{n-1}$ and there are exponentially many examples showing that this bound is tight.

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1 Introduction

Let X be a finite set and let 2^X be the system of all subsets of X . The basic problem of the theory of extremal sets systems is to determine the maximum size that a set system $\mathcal{F} \subseteq 2^X$ can have provided \mathcal{F} satisfies a prescribed property. The prototypes of investigated properties are the intersecting and Sperner properties. A set system \mathcal{F} is *intersecting* if $F_1 \cap F_2 \neq \emptyset$ for any pair $F_1, F_2 \in \mathcal{F}$ and a set system \mathcal{F} is *Sperner* if there do not exist two distinct sets $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subset F_2$. The celebrated theorems of Erdős, Ko, Rado [4] and of Sperner [13] determine the largest size that a uniform intersecting set system and Sperner system can have. Both theorems have many applications and generalizations.

One such generalization of the Sperner property is the so called *more part Sperner property*. In this case, the underlying set X is partitioned into m subsets X_1, \dots, X_m and the system $\mathcal{F} \subseteq 2^X$ is said to be *m-part Sperner* if for any pair $F_1, F_2 \in \mathcal{F}$ with $F_1 \subset F_2$ there exist at least two indices $1 \leq i_1 < i_2 \leq m$ such that $F_1 \cap X_{i_j} \subsetneq F_2 \cap X_{i_j}$ holds for $j = 1, 2$. Systems with this property were first considered in [9, 11]; for a survey of recent results see [2].

In this paper we will consider analogous problems for intersection properties and also some mixed more part properties in the case when m equals 2. All maximum size 2-part Sperner set systems were described by P.L. Erdős and G.O.H. Katona in [5, 6]. To rephrase the 2-part Sperner property it is convenient to introduce the following set systems of *traces*: for any $A \subseteq X_1$ and $B \subseteq X_2$ let $\mathcal{F}_A = \{F \cap X_2 : F \in \mathcal{F}, F \cap X_1 = A\}$, $\mathcal{F}_B = \{F \cap X_1 : F \in \mathcal{F}, F \cap X_2 = B\}$. Also, for any $F \in \mathcal{F}$ we will call $F \cap X_1$ and $F \cap X_2$ the *traces* of F on X_1 and X_2 . One can easily see that a set system \mathcal{F} is 2-part Sperner with respect to the partition $X = X_1 \cup X_2$ if and only if for any subset $A \subseteq X_1$ or $B \subseteq X_2$ the set systems \mathcal{F}_A and \mathcal{F}_B possess the Sperner property.

Having this equivalence in mind, it is natural to introduce the following three definitions where we always assume that the underlying set X is partitioned into two sets X_1 and X_2 :

Definition 1. (i) a set system $\mathcal{F} \subseteq 2^X$ is 2-part intersecting (a 2I-system for short) if for any subset A of X_1 (and for any subset B of X_2) the trace system \mathcal{F}_A on X_2 (and the trace system \mathcal{F}_B on X_1) is intersecting,

(ii) a set system $\mathcal{F} \subseteq 2^X$ is 2-part intersecting, 2-part Sperner (a 2I2S-system for short) if for any subset A of X_1 and for any subset B of X_2 the trace systems \mathcal{F}_A on X_2 and \mathcal{F}_B on X_1 are intersecting and Sperner,

(iii) a set system $\mathcal{F} \subseteq 2^X$ is 1-part intersecting, 1-part Sperner (a 1I1S-system for short) if there exists no pair of distinct sets F_1, F_2 in \mathcal{F} such that the traces of F_1, F_2 are disjoint at one of the parts and are in containment at the other.

We will address the problem of finding the maximum possible size of a set system possessing the properties above. Some of our bounds will apply regardless of the sizes of the parts in the 2-partition and some will only apply to special cases. We will be mostly interested in the case when $|X_1| = |X_2|$. Clearly, for any 2-part set system \mathcal{F} we have $|\mathcal{F}| = \sum_{A \subseteq X_1} |\mathcal{F}_A| = \sum_{B \subseteq X_2} |\mathcal{F}_B|$. As any intersecting system of subsets of X_1 has size at most $2^{|X_1|-1}$, it follows that any 2I-system has size at most $2^{|X_2|}2^{|X_1|-1} = 2^{|X|-1}$. In Section 2 we will prove the following theorem.

Theorem 2. *Let $\mathcal{F} \subseteq 2^X$ be a 2-part intersecting system of maximum size. If the 2-partition $X = X_1 \cup X_2$ is non-trivial (i.e. $X_1 \neq \emptyset$, $X_2 \neq \emptyset$), then the following inequality holds:*

$$|\mathcal{F}| \leq \frac{3}{8}2^{|X|}.$$

The bound is best possible if X_1 or X_2 is a singleton. Moreover, if $|X_1| = |X_2|$, then there exists a 2-part intersecting system of size $\frac{1}{3}(2^{|X|} + 2)$.

The rest of Section 2 is devoted to 2I2S systems. We prove the following result.

Theorem 3. *Let $\mathcal{F} \subseteq 2^X$ be a 2-part intersecting, 2-part Sperner system of maximum size. Then $|\mathcal{F}| \leq \binom{|X|}{\lfloor |X|/2 \rfloor}$ holds. This bound is asymptotically sharp as long as $|X_1|^{1/2} = o(|X_2|)$. If $|X_1| = |X_2|$ holds, then there exists a 2I2S system of size $c \binom{|X|}{\lfloor |X|/2 \rfloor}$ with $c > 2/3$.*

The main result of the paper is proved in Section 3. We determine the maximum size of a 1-part intersecting 1-part Sperner set system.

Theorem 4. *Let \mathcal{F} be a maximum size 1-part intersecting, 1-part Sperner set system. Then $|\mathcal{F}| = 2^{|X|-2}$.*

2 2I- and 2I2S-systems

In this section we consider two-part intersecting and two-part intersecting, two-part Sperner set systems. We first consider a general construction that produces large families with these properties. Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ and $\mathcal{B}_1, \dots, \mathcal{B}_m$ be partitions of 2^{X_1} and 2^{X_2} into disjoint intersecting (intersecting, Sperner) systems some of which may possibly be empty. Then the set system $\mathcal{F} := \cup_{i=1}^m \mathcal{A}_i \times \mathcal{B}_i = \{A \cup B : A \in \mathcal{A}_i, B \in \mathcal{B}_i \text{ for some } 1 \leq i \leq m\}$ is a 2I-(2I2S)-system by definition.

Fact 5. Let $0 \leq x_1 \leq \dots \leq x_n$, $0 \leq y_1 \leq \dots \leq y_n$ be real numbers and π be a permutation of the first n integers. Then we have the following inequalities:

$$\sum_{i=1}^n x_i y_{\pi(i)} \leq \sum_{i=1}^n x_i y_i \leq \max \left\{ \sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2 \right\}.$$

Thus to maximize the size of a family obtained through the general construction one should enumerate the \mathcal{A}_i 's and the \mathcal{B}_i 's in decreasing order according to their size. Moreover, if $|X_1| = |X_2|$, then it is enough to consider partitions $\mathcal{A}_1, \dots, \mathcal{A}_m$ of 2^{X_1} and the sum $\sum_{i=1}^m |\mathcal{A}_i|^2$.

2.1 Two-part intersecting systems

In this subsection we prove Theorem 2. In the proof we use the following theorem of Kleitman [10].

Theorem 6 (Kleitman [10]). *Let $\mathcal{F}_1, \dots, \mathcal{F}_m \subseteq 2^{[n]}$ be intersecting set systems. Then*

$$|\mathcal{F}_1 \cup \dots \cup \mathcal{F}_m| \leq 2^n - 2^{n-m}.$$

Proof of Theorem 2. For any subset A of X_1 let \bar{A} denote its complement $X_1 \setminus A$. By definition, both \mathcal{F}_A and $\mathcal{F}_{\bar{A}}$ are intersecting. Also, these set systems are disjoint as $B \in \mathcal{F}_A \cap \mathcal{F}_{\bar{A}}$ implies $B \cup A, B \cup \bar{A} \in \mathcal{F}$ which contradicts the 2-part intersecting property of \mathcal{F} . Thus by Theorem 6 we have $|\mathcal{F}_A| + |\mathcal{F}_{\bar{A}}| \leq 2^{|X_2|-1} + 2^{|X_2|-2}$.

Altogether we obtain

$$|\mathcal{F}| \leq 2^{|X_1|-1}(2^{|X_2|-1} + 2^{|X_2|-2}) = \frac{3}{8}2^{|X|}.$$

Our best lower bounds arise from our general construction. If X_1 consists of a single element x_1 , then let $\mathcal{A}_1 = \{\{x_1\}\}$, $\mathcal{A}_2 = \{\emptyset\}$ and $\mathcal{B}_1 = \{B \subset X_2 : x_2 \in B\}$, $\mathcal{B}_2 = \{B \subset X_2 : x_2 \notin B, x'_2 \in B\}$ for two fixed elements $x_2, x'_2 \in X_2$ and the other \mathcal{B}_i 's be arbitrary while the other \mathcal{A}_i 's be empty. For the set system \mathcal{F} we obtain via the general construction, we have $|\mathcal{F}| = 2^{|X_2|-1} + 2^{|X_2|-2} = \frac{3}{8}2^{|X|}$.

Finally, let us suppose that $|X_1| = |X_2| = |X|/2$ and let the elements of X_1 and X_2 be x_1^1, \dots, x_m^1 and x_1^2, \dots, x_m^2 . Let us define the partition of 2^{X_1} and 2^{X_2} in the following way: $\mathcal{A}_i := \{A \subset X_1 \setminus \{x_1^1, \dots, x_{i-1}^1\} : x_i^1 \in A\}$, $\mathcal{B}_i := \{B \subset X_2 \setminus \{x_1^2, \dots, x_{i-1}^2\} : x_i^2 \in B\}$ for all $1 \leq i \leq m+1$ (i.e. $\mathcal{A}_{m+1} = \mathcal{B}_{m+1} = \{\emptyset\}$). Then for the set system \mathcal{F} arising from the general construction we have

$$|\mathcal{F}| = 1 + \sum_{i=1}^m 2^{|X|-2i} = \frac{2^{|X|} + 2}{3}. \quad \square$$

Remark 7. Theorem 6 shows that the above set system for the $|X_1| = |X_2|$ case is best possible among those that we can obtain via the general construction. Indeed, by Fact 5 we know that we have to consider partitions of 2^{X_1} to intersecting set systems with sizes s_1, s_2, \dots, s_m and maximize $\sum_{i=1}^m s_i^2$. But a partition maximizes this sum of squares if for all $1 \leq j \leq m$ the sums $\sum_{i=1}^j s_i$ are maximized. In the construction we use, the sums $\sum_{i=1}^j s_i$ match the upper bound of Theorem 6.

2.2 Two-part intersecting, two-part Sperner systems

In this subsection we consider 2I2S-systems and prove Theorem 3. To be able to use the general construction, we need to define a partition of the power set into intersecting Sperner set systems.

Construction 8. Here we give a partition of the power set of Y into intersecting Sperner systems where all levels are partitioned into minimal number of (uniform) intersecting systems (we call this *canonical partition*). This partition is in the form of

$$\begin{aligned} \mathcal{Y}_k, & \quad \text{for } k = \left\lceil \frac{|Y|+1}{2} \right\rceil, \dots, |Y|; \\ \mathcal{Y}_{i,j}, & \quad \text{for } i = 1, \dots, \left\lceil \frac{|Y|+1}{2} \right\rceil - 1, j = 1, \dots, |Y| - 2i + 1; \\ \mathcal{Y}_\ell^*, & \quad \text{for } \ell = 0, \dots, \left\lceil \frac{|Y|+1}{2} \right\rceil - 1. \end{aligned}$$

The systems \mathcal{Y}_k are $\binom{Y}{k}$. Fix an enumeration $y_1, \dots, y_{|Y|}$ of the elements of Y and define the systems $\mathcal{Y}_{i,j}$ as $\left\{ Y' \in \binom{Y \setminus \{y_1, \dots, y_{j-1}\}}{i} : y_j \in Y' \right\}$. Finally let $\mathcal{Y}_\ell^* = \binom{Y}{\ell} \setminus \bigcup_{j=1}^{|Y|-2\ell+1} \mathcal{Y}_{\ell,j}$. We remark that the second and third types are identical to those in the corresponding Kneser construction. Note that the number of systems in the partition is quadratic in $|Y|$ but for any $\varepsilon > 0$ a linear number of the systems contain $(1 - \varepsilon)$ fraction of all sets.

Proof of Theorem 3. The upper bound of the theorem follows from the result of Katona [9] and Kleitman [11] stating that a 2-part Sperner system has size at most $\binom{|X|}{\lfloor |X|/2 \rfloor}$, since any 2I2S-system is 2-part Sperner.

We now prove the lower bound. First we consider the case when the size of X_1 is negligible compared to the size of X_2 . Let us assume that $|X_1| = o(|X_2|^{1/2})$. Then $o(|X_2|^{1/2})$ of the set systems in the canonical partition of 2^{X_1} contain $(1 - \varepsilon)$ fraction of 2^{X_1} . If $i = o(|X_2|^{1/2})$, then the system $\binom{X_2}{\lfloor |X_2|/2 + i}$ is intersecting Sperner and has size $(1 - o(1)) \binom{|X_2|}{\lfloor |X_2|/2} = (1/2)^{|X_1|} - o(1) \binom{|X|}{\lfloor |X|/2}$ and thus the system \mathcal{F} arising from this pair of partitions has size $(1 - o(1)) \binom{|X|}{\lfloor |X|/2}$.

In the case when $|X_1| = |X_2|$, then we first show that the 2I2S-system \mathcal{F} we derive from the canonical partition using our general construction has size $(2/3 - o(1)) \binom{|X|}{\lfloor |X|/2}$. We then use Frankl and Füredi's construction [7] to improve this bound by a constant factor. For sake of simplicity, assume $n = |X|$ is divisible by 4. Then our system has size

$$\sum_{i=n/4+1}^{n/2} \binom{n/2}{i}^2 + \sum_{i=1}^{n/4} \sum_{k=0}^{n/2-2i} \binom{n-1-k}{i-1}^2 + \sum_{i=1}^{n/4} \binom{2i-1}{i}^2,$$

where the sums belong to the three different system types in the canonical partition. We can write our system as $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where the first subsystem corresponds to the sets listed in the first summation, while the second one consists of the other elements. Then one can see that the cardinality of \mathcal{F}_1 is

$$\begin{aligned} \sum_{i=n/4+1}^{n/2} \binom{n/2}{i}^2 &= \sum_{i=n/4+1}^{n/2} \binom{n/2}{i} \binom{n/2}{n/2-i} \\ &= 1/2 \binom{n}{n/2} - \binom{n/2}{n/4}^2 = (1/2 - o(1)) \binom{n}{n/2} \end{aligned}$$

as $\binom{n/2}{i} \binom{n/2}{n/2-i}$ is the number of those $n/2$ -subsets of X that intersect X_1 in i elements.

Next we prove that $|\mathcal{F}_2|$ is roughly $\frac{1}{3}|\mathcal{F}_1|$ which proves that $|\mathcal{F}_2|$ is $\frac{1/2-o(1)}{3} \binom{n}{n/2}$ and thus $|\mathcal{F}| = (\frac{2}{3} - o(1)) \binom{n}{n/2}$. We consider those members of \mathcal{F}_2 which intersect X_1 in i elements (and then intersect X_2 in i elements too). We will show that the number of these sets is roughly the third of the number of those members of \mathcal{F}_1 , which intersect X_1 (and then X_2 as well) in $n/2 - i$ elements. We compare

$$\binom{2i-1}{i}^2 + \sum_{k=0}^{n/2-2i} \binom{n/2-1-k}{i-1}^2 \quad \text{to} \quad \binom{n/2}{n/2-i}^2 = \binom{n/2}{i}^2.$$

One can easily see that for all possible values

$$\begin{aligned} \binom{n/2-1-k}{i-1} &\leq 2^{-k} \binom{n/2}{i} \quad \text{and} \\ \binom{2i-1}{i} &= \binom{n/2-1-(n/2-2i)}{i-1} \leq 2^{2i-n/2} \binom{n/2}{i}. \end{aligned}$$

This implies

$$\binom{2i-1}{i}^2 + \sum_{k=0}^{n/2-2i} \binom{n/2-1-k}{i-1}^2 \leq \left(2^{4i-n} + \sum_{k=0}^{n/2-2i} 2^{-2k} \right) \binom{n/2}{i}^2.$$

If $n/2 - 2i$ goes to infinity, $2^{4i-n} + \sum_{k=0}^{n/2-2i} 2^{-2k}$ goes to $1/3$, which means that for any ϵ it is at most $1/3 + \epsilon$ with the exception of c consecutive levels, where c depends only on ϵ and not on n . If n is large enough, those c levels do not matter too much, hence $|\mathcal{F}_2| \leq (\frac{1}{3} + o(1))|\mathcal{F}_1|$.

On the other hand, since $1 \leq i \leq n/4$, we have

$$\frac{\binom{n/2-1}{i-1}}{\binom{n/2}{i}} = \frac{i}{n/2} \quad \text{and} \quad \frac{\binom{m-1}{i-1}}{\binom{m}{i-1}} = \frac{m-i+1}{m} \geq 1/2,$$

where m is always at least $2i$. This, in turn, implies that

$$\binom{n/2 - 1 - k}{i - 1} \geq 2^{-k} \frac{i}{n/2} \binom{n/2}{i}$$

and then

$$\sum_{k=0}^{\frac{n}{2}-2i} \binom{n/2 - 1 - k}{i - 1}^2 \geq \left(\frac{i}{n/2}\right)^2 \sum_{k=0}^{\frac{n}{2}-2i} 2^{-2k} \binom{n/2}{i}^2 \geq \frac{4}{3} \left(\frac{i}{n/2}\right)^2 \binom{n/2}{i}^2.$$

Thus the ratio we want to find is at least $1/3 - \varepsilon$ for a small $\varepsilon > 0$ if $\frac{i}{n/2} \geq 1/2 - \varepsilon$. It is true for the largest $\varepsilon n/2$ levels, and that is enough as the number of the other elements of \mathcal{F}_2 is far too small.

Frankl and Füredi considered in [7] the following pair of k -uniform intersecting set systems on a base set Y : let Y be equipartitioned into $Y_1 \cup Y_2$ and define

$$\mathcal{G}_1^i = \left\{ G \in \binom{Y}{i} : |Y_1 \cap G| > |Y_1|/2 \right\},$$

$$\mathcal{G}_2^i = \left\{ G \in \binom{Y}{i} \setminus \mathcal{G}_1^i : |Y_2 \cap G| > |Y_2|/2 \right\}.$$

They observed that if $|Y| = 2i + o(i^{1/2})$, then $|\mathcal{G}_1^i \cup \mathcal{G}_2^i| = (1 - o(1)) \binom{|Y|}{i}$ and that for any $\alpha > 0$ there exists $\beta > 0$ such that if $|Y| \leq 2i + \beta i^{1/2}$, then $|\mathcal{G}_1^i \cup \mathcal{G}_2^i| \geq (1 - \alpha) \binom{|Y|}{i}$.

Let us fix $0 < \alpha < 1/6$ and consider β as above. We define a modified version of the canonical partition for a given set Y . We replace the set systems $\mathcal{Y}_{i,j}$ for all $\frac{|Y|}{2} - \frac{\delta}{2}|Y|^{1/2} \leq i \leq \frac{|Y|}{2}$ and $j = 1, \dots, |Y| - 2i + 1$ with \mathcal{G}_1^i and \mathcal{G}_2^i . Observe that the ratio of $|\mathcal{G}_1^i|^2 + |\mathcal{G}_2^i|^2$ and $\binom{n}{i}^2$ is $1/2 - \alpha$ instead of $1/3$. Furthermore, note that for any $\beta > 0$ the sum $\sum_{i=n/4-\delta n^{1/2}}^{n/4} \binom{n/2}{i}^2$ is a positive, say γ -fraction of $\sum_{i=0}^{n/4} \binom{n/2}{i}^2$. Thus with the above replacement we made \mathcal{F}_2 larger by a factor of $1 + \frac{\gamma(1-6\alpha)}{2}$. \square

Katona's proof that a 2-part Sperner set system can contain at most $\binom{n}{\lfloor n/2 \rfloor}$ sets used a theorem of Erdős [3] on the number of sets contained in the union of k Sperner set systems. Our proofs of Theorem 2 and Remark 7 used Theorem 6, Kleitman's result on the size of the union of k intersecting families. It seems natural to ask how large can the union of k intersecting Sperner set systems be as the problem seems to be interesting on its own right and it might help in establishing bounds on 2S2I-systems. Unfortunately, we were only able to determine the exact result in the very special case when $k = 2$ and n is odd. The result follows easily from the following theorem of Greene, Katona and Kleitman.

Theorem 9 (Greene, Katona, Kleitman [8]). *If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting and Sperner set system, then the following inequality holds*

$$\sum_{F \in \mathcal{F}, |F| \leq n/2} \frac{1}{\binom{n}{|F|-1}} + \sum_{F \in \mathcal{F}, |F| > n/2} \frac{1}{\binom{n}{|F|}} \leq 1.$$

Corollary 10. *Let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ be intersecting Sperner set systems and $n = 2l + 1$ an odd integer. Then we have $|\mathcal{F} \cup \mathcal{G}| \leq \binom{n}{l+1} + \binom{n}{l+2}$ and the inequality is sharp as shown by $\mathcal{F} = \binom{[n]}{l+1}, \mathcal{G} = \binom{[n]}{l+2}$.*

Proof. We may assume that \mathcal{F} and \mathcal{G} are disjoint. Let us add the inequality of Theorem 9 for both systems \mathcal{F} and \mathcal{G} . The bigger the number of the summands, the greater the cardinality of \mathcal{F} , therefore we need to keep the summands as small as possible to obtain the greatest number of summands. The set size for which the summand is the smallest is $l + 1$ and the second smallest summand is for set sizes l and $l + 2$. As by the disjointness of the systems the number of smallest summands is at most $\binom{n}{l+1}$, the result follows. \square

3 1-part intersecting, 1-part Sperner systems

In this section we study 1-part Sperner 1-part intersecting set systems and prove Theorem 4. In order to prove the result we need a further definition. We say that the set systems \mathcal{F} and \mathcal{G} are *intersecting, cross-Sperner* if both \mathcal{F} and \mathcal{G} are intersecting and there is no $F \in \mathcal{F}, G \in \mathcal{G}$ with $F \subset G$ or $G \subset F$. We will prove the following theorem which can be of independent interest.

Theorem 11. *Let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ be a pair of cross-Sperner, intersecting set systems. Then we have*

$$|\mathcal{F}| + |\mathcal{G}| \leq 2^{n-1}$$

and this bound is best possible.

One of our main tools will be the following special case of the Four Functions Theorem of Ahlswede and Daykin [1]. Let us write $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

Theorem 12 (Ahlswede-Daykin, [1]). *For any pair \mathcal{A}, \mathcal{B} of set systems we have*

$$|\mathcal{A}||\mathcal{B}| \leq |\mathcal{A} \wedge \mathcal{B}||\mathcal{A} \vee \mathcal{B}|.$$

The other result we will use in our argument is due to Marica and Schönheim [12] and involves the difference set system $\Delta(\mathcal{F}) = \{F \setminus F' : F, F' \in \mathcal{F}\}$.

Theorem 13 (Marica – Schönheim [12]). For any set system \mathcal{F} we have $|\Delta(\mathcal{F})| \geq |\mathcal{F}|$.

Corollary 14. Let \mathcal{D} be a downward closed set system and let \mathcal{F} be an intersecting subsystem of \mathcal{D} . Then the inequality $2|\mathcal{F}| \leq |\mathcal{D}|$ holds.

Proof. As \mathcal{D} is downward closed and $\mathcal{F} \subset \mathcal{D}$, it follows that $\Delta(\mathcal{F}) \subset \mathcal{D}$. Furthermore, as \mathcal{F} is intersecting, we have $\mathcal{F} \cap \Delta(\mathcal{F}) = \emptyset$ and thus we are done by Theorem 13. \square

Proof of Theorem 11. Let us begin with defining the following four set systems

$$\mathcal{U} = \{U \subseteq [n] : \exists H \in \mathcal{F} \cup \mathcal{G} \text{ such that } H \subseteq U\}, \quad \mathcal{U}' = \mathcal{U} \setminus (\mathcal{F} \cup \mathcal{G}),$$

$$\mathcal{D} = \{D \subseteq [n] : \exists H \in \mathcal{F} \cup \mathcal{G} \text{ such that } D \subseteq H\}, \quad \mathcal{D}' = \mathcal{D} \setminus (\mathcal{F} \cup \mathcal{G}).$$

Clearly, $\mathcal{D}'' = \{D' : \exists F \in \mathcal{F} \text{ such that } D' \subset F\}$ is downward closed (and, by definition, $\mathcal{F} \subset \mathcal{D}''$), hence by Corollary 14 we have $2|\mathcal{F}| \leq |\mathcal{D}''|$. Moreover by the cross-Sperner property, we have $(\mathcal{D}'' \setminus \mathcal{F}) \cap \mathcal{G} = \emptyset$, and therefore we have $\mathcal{D}'' \setminus \mathcal{F} \subset \mathcal{D}'$. Consequently $|\mathcal{F}| \leq |\mathcal{D}'|$ and, by symmetry, $|\mathcal{G}| \leq |\mathcal{D}'|$ also holds.

Note that $\mathcal{F} \wedge \mathcal{G} \subset \mathcal{D}'$. Indeed, $F \cap G \in \mathcal{D}$ by definition and $F \cap G \in \mathcal{F}$ (or $F \cap G \in \mathcal{G}$) would contradict the cross-Sperner property. Similarly, we obtain that $\mathcal{F} \vee \mathcal{G} \subset \mathcal{U}'$ and it is easy to see that the cross-Sperner property implies that $\mathcal{U}' \cap \mathcal{D}' = \emptyset$ and thus the four systems $\mathcal{F}, \mathcal{G}, \mathcal{U}', \mathcal{D}'$ are pairwise disjoint.

Now suppose as a contradiction that $|\mathcal{F}| + |\mathcal{G}| > 2^{n-1}$ and thus $|\mathcal{U}'| + |\mathcal{D}'| < 2^{n-1}$. By $|\mathcal{F}|, |\mathcal{G}| \leq |\mathcal{D}'|$ we obtain that $|\mathcal{U}'| < |\mathcal{F}|, |\mathcal{G}|$ and thus using Theorem 12 we have

$$|\mathcal{U}'||\mathcal{D}'| < |\mathcal{F}||\mathcal{G}| \leq |\mathcal{F} \wedge \mathcal{G}||\mathcal{F} \vee \mathcal{G}| \leq |\mathcal{U}'||\mathcal{D}'|,$$

a contradiction.

Finally, let us mention some pairs of set systems for which the sum of their sizes equals 2^{n-1} . Any maximum intersecting system \mathcal{F} with \mathcal{G} the empty set system is extremal, just as the pair $\mathcal{F}_1 = \{F \subset [n] : 1 \in F, 2 \notin F\}$, $\mathcal{G}_1 = \{G \subset [n] : 1 \notin G, 2 \in G\}$. Furthermore, for any $k \geq n/2$ the pair $\mathcal{F}_k = \{F \subset [n] : 1 \in F, |F| \leq k\}$, $\mathcal{G}_k = \{G \subset [n] : 1 \notin G, |G| \geq k\}$ has the required property, too. \square

Proof of Theorem 4. First let us consider any pair of maximal intersecting systems $\mathcal{A} \subseteq 2^{X_1}$, $\mathcal{B} \subseteq 2^{X_2}$. Clearly, the set system $\mathcal{F} = \mathcal{A} \times \mathcal{B}$ is a 1IIS-system as any pair of sets $F_1, F_2 \in \mathcal{F}$ intersect both in X_1 and in X_2 . This shows that a maximum 1IIS-system contains at least $2^{|X|-2}$ sets.

To obtain the upper bound of the theorem let \mathcal{F} be any 1IIS-system. For any $A \subseteq X_1$ let \bar{A} denote $X_1 \setminus A$. By definition, both \mathcal{F}_A and $\mathcal{F}_{\bar{A}}$ are intersecting systems, and no element of the first can contain any element of the second (and vice versa). In other words they form a pair of intersecting, cross-Sperner systems. Due to Theorem 11 we have $|\mathcal{F}_A| + |\mathcal{F}_{\bar{A}}| \leq 2^{|X_2|-1}$. The number of pairs A, \bar{A} is $2^{|X_1|-1}$ therefore we have $|\mathcal{F}| \leq 2^{|X|-2}$. \square

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